

THE CLASSIFICATION OF HOMOGENEOUS EINSTEIN METRICS ON FLAG MANIFOLDS WITH $b_2(M) = 1$

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ABSTRACT. Let G be a simple compact connected Lie group. We study homogeneous Einstein metrics for a class of compact homogeneous spaces, namely generalized flag manifolds G/H with second Betti number $b_2(G/H) = 1$. There are 8 infinite families G/H corresponding to a classical simple Lie group G and 25 exceptional flag manifolds, which all have some common geometric features; for example they admit a unique invariant complex structure which gives rise to unique invariant Kähler–Einstein metric. The most typical examples are the compact isotropy irreducible Hermitian symmetric spaces for which the Killing form is the unique homogeneous Einstein metric (which is Kähler). For non-isotropy irreducible spaces the classification of homogeneous Einstein metrics has been completed for 24 of the 26 cases. In this paper we construct the Einstein equation for the two unexamined cases, namely the flag manifolds $E_8/U(1) \times SU(4) \times SU(5)$ and $E_8/U(1) \times SU(2) \times SU(3) \times SU(5)$. In order to determine explicitly the Ricci tensors of an E_8 -invariant metric we use a method based on the Riemannian submersions. For both spaces we classify all homogeneous Einstein metrics and thus we conclude that any flag manifold G/H with $b_2(M) = 1$ admits a finite number of non-isometric non-Kähler invariant Einstein metrics. The precise number of these metrics is given in Table 1.

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INTRODUCTION

Given a Riemannian manifold M the question whether M carries an Einstein metric, that is a Riemannian metric g of constant Ricci curvature, is a fundamental one in Riemannian geometry. The Einstein equation $\text{Ric}_g = \lambda \cdot g$ ($\lambda \in \mathbb{R}$) reduces to a system of a non-linear second order PDEs and a good understanding of its solutions in the general case seems far from being attained. If M is compact, then Einstein metrics (of volume 1) become in a natural way privileged metrics since they are characterized variationally as the critical points of the total scalar curvature functional $T : \mathcal{M} \rightarrow \mathbb{R}$, given by $T(g) = \int_M S_g dV_g$, restricted to the set \mathcal{M}_1 of Riemannian metrics of volume 1. However, even in this case general existence results are difficult to obtain. If we consider a homogeneous G -space $M = G/H$, then it is natural to work with G -invariant Riemannian metrics. For such a metric the Einstein equation reduces to an algebraic system which is more manageable and in some cases it can be solved explicitly. Most known examples of Einstein manifolds are homogeneous.

A generalized flag manifold is an adjoint orbit $M = \text{Ad}(G)w$ ($w \in \mathfrak{g}$) of a compact connected semi-simple Lie group G and it can be represented as a compact homogeneous space of the form $M = G/H = G/C(S)$, where $C(S)$ is the centralizer of a torus S in G (and thus $\text{rk } G = \text{rk } H$). Generalized flag manifolds have been classified in terms of painted Dynkin diagrams and these have Kähler metrics, that is, the homogeneous manifolds $M = G/H$ can be expressed as $G^\mathbb{C}/U$ where $G^\mathbb{C}$ is the complexification of G and U a parabolic subgroup of $G^\mathbb{C}$. It is also known that there are a finite number of invariant complex structures on M and for each complex structure there is a compatible G -invariant Kähler–Einstein metric. In this paper we investigate invariant Einstein metrics on generalized flag manifolds $M = G/H$ of a compact connected simple Lie group G with second Betti number $b_2(M) = 1$. Such a space is determined by painting black in the Dynkin diagram of G only one simple root. By [BHi] it is known that $M = G/H$ admits a unique invariant complex structure, and thus a unique Kähler–Einstein metric. Compact irreducible Hermitian symmetric spaces are the most typical examples of this category, and these are the only flag manifolds for which the Kähler–Einstein metric is given by the Killing form. Generalized flag manifolds $M = G/H$ with $b_2(M) = 1$ can be divided into following six classes, with respect to the height of the painted black simple root (see §3), or equivalently, with respect to the decomposition of the associated isotropy representation (see Table 1):

(A) The compact irreducible Hermitian symmetric spaces $M = G/H$, which admit (up to scaling) a unique invariant Einstein metric. In this case the height of the painted black simple root is equal to 1.

(B) The flag manifolds $M = G/H$ for which the isotropy representation decomposes into two inequivalent irreducible $\text{Ad}(H)$ -submodules, i.e., $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. These spaces are determined by painting black a simple root with height 2 and their classification was obtained in [AC1] (see also [Sak]).

(C) Seven flag manifolds $M = G/H$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$. These spaces were determined by painting black a simple root with height 3 [Kim].

(D) Four flag manifolds $M = G/H$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$. These spaces are determined by painting black a simple root with height 4 [AC3].

(E) The flag manifold $M = G/H = E_8 / U(1) \times SU(4) \times SU(5)$. It is determined by painting black the simple root α_4 and its isotropy representation is such that with $\mathfrak{m} = \mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_5$.

(F) The flag manifold $M = G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$. It is determined by painting black the simple root α_5 and the associated isotropy representation is such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_6$.

Table 1. The number $\mathcal{E}(M)$ of non-isometric invariant Einstein metrics on generalized flag manifolds with $b_2(M) = 1$.

$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$	$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$
(A) Hermitian Symmetric Spaces ([Wo1])		(C) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ ([Kim], [AnC])	
$SU(\ell)/S(U(p) \times U(\ell - p))$	= 1	$F_4 / U(3) \times SU(2)$	= 3
$SO(2\ell + 1)/SO(2) \times SO(2\ell - 1)$	= 1	$E_6 / U(2) \times SU(3) \times SU(3)$	= 3
$Sp(\ell)/U(\ell)$	= 1	$E_7 / U(3) \times SU(5)$	= 3
$SO(2\ell)/SO(2) \times SO(2\ell - 2)$	= 1	$E_7 / SU(2) \times SU(6) \times U(1)$	= 3
$SO(2\ell)/U(\ell)$	= 1	$E_8 / E_6 \times SU(2) \times U(1)$	= 3
$E_6 / U(1) \times SO(10)$	= 1	$E_8 / U(8)$	= 3
$E_7 / U(1) \times E_6$	= 1	$G_2 / U(2)$ ($U(2)$ represented by the long root)	= 3
(B) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ ([DiK], [AC2])		(D) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$ ([AC3])	
$SO(2\ell + 1)/U(\ell - m) \times SO(2m + 1)$ ($\ell - m \neq 1$)	= 2	$F_4 / SU(3) \times SU(2) \times U(1)$	= 3
$Sp(\ell)/U(\ell - m) \times Sp(m)$ ($m \neq 0$)	= 2	$E_7 / SU(4) \times SU(3) \times SU(2) \times U(1)$	= 3
$SO(2\ell)/U(\ell - m) \times SO(2m)$ ($\ell - m \neq 1, m \neq 0$)	= 2	$E_8 / SU(7) \times SU(2) \times U(1)$	= 3
$G_2 / U(2)$ ($U(2)$ represented by the short root)	= 2	$E_8 / SO(10) \times SU(3) \times U(1)$	= 5
$F_4 / SO(7) \times U(1)$	= 2	(E) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$	
$F_4 / Sp(3) \times U(1)$	= 2	$E_8 / SU(5) \times SU(4) \times U(1)$	= 6 (new)
$E_6 / SU(6) \times U(1)$	= 2	(F) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$	
$E_6 / SU(2) \times SU(5) \times U(1)$	= 2	$E_8 / SU(5) \times SU(3) \times SU(2) \times U(1)$	= 5 (new)
$E_7 / SU(7) \times U(1)$	= 2		
$E_7 / SU(2) \times SO(10) \times U(1)$	= 2		
$E_7 / SO(12) \times U(1)$	= 2		
$E_8 / E_7 \times U(1)$	= 2		
$E_8 / SO(14) \times U(1)$	= 2		

As one can see in Table 1, homogeneous Einstein metrics of the first four classes (A)-(D) have been completely classified in [Sak], [DiK], [AC2], [Kim] and [AC3] (see also the recent work [AnC], where invariant Einstein metrics were studied under the more general context of Ricci flow). In particular, only the cases (E) and (F) have not been examined yet. In this article we focus on these two flag manifolds and by applying a method based on the Riemannian submersions we construct the homogeneous Einstein equation. Moreover for both cases we manage to classify all (non-isometric) homogeneous Einstein metrics. Our main results are stated as follows:

Theorem A. *The generalized flag manifold $M = G/H = E_8 / U(1) \times SU(4) \times SU(5)$ admits (up to an isometry and a scale) precisely five non-Kähler E_8 -invariant Einstein metrics (see Theorem 2).*

Theorem B. *The generalized flag manifold $M = G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ admits (up to an isometry and a scale) precisely four non-Kähler E_8 -invariant Einstein metrics (see Theorem 3).*

Notice that the construction as well as the determination of all real positive solutions of the homogeneous Einstein equation on $E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$, is much more complicated than case (E). For example here we find 9 non-zero structure constants with respect to the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$ (see also [Chr]). In order to determine them explicitly we use the method of Riemannian submersions as well as the method based on the twistor fibration of G/H over the symmetric space $E_8 / E_7 \times SU(2)$, a method which was initially presented in the first author Phd's thesis ([AC3]). For this space the system of algebraic equations which give the homogeneous Einstein equation consists of five non-linear polynomial equations and it seems that it is difficult to compute a Gröbner basis. However we are able to obtain all positive real solutions approximately by using the software package HOM4PS-2.0, which implements the polyhedral homotopy continuation method for solving polynomial systems of equations with several variables ([LeLT]).

From previous results of Einstein metrics on flag manifolds G/H with $b_2(G/H) = 1$ and Theorems A and B, we conclude that

Main Theorem. *Let G a compact connected simple Lie group and let $M = G/H$ be a generalized flag manifold with first Betti number $b_2(G/H) = 1$, which is not an irreducible Hermitian symmetric space of compact type. Then M admits a finite number of non-isometric G -invariant Einstein metrics which are not Kähler.*

It is worth to mention that the results of this work support the finiteness conjecture of invariant Einstein metrics on reductive homogeneous spaces G/H with simple spectrum (cf. [BWZ]).

The paper is organized as follows: We describe the Ricci tensor on a reductive homogeneous space in §1 and Riemannian submersions of homogeneous spaces in §2, and in §3 we discuss the algebraic setting of flag manifolds. In §4 we treat the space $M = G/H = E_8 / U(1) \times SU(4) \times SU(5)$, we write down explicitly the homogeneous Einstein equation and we prove Theorem A. For the second flag manifold $M = G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ and Theorem B, this will be done in §5.

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1. THE RICCI TENSOR OF A G -INVARIANT METRIC

Let G be a compact connected semi-simple Lie group with Lie algebra \mathfrak{g} , and let H be a closed subgroup of G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. We denote by B the negative of the Killing form of \mathfrak{g} . Then B is an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} . Let \mathfrak{m} be an $\text{Ad}(H)$ -invariant orthogonal complement of \mathfrak{h} with respect to B , that means $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$. As usual we identify $\mathfrak{m} = T_o G/H$, where $o = eH \in G/H$. We assume that $\mathfrak{m} = T_o G/H$ admits a decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ into q irreducible $\text{Ad}(H)$ -modules \mathfrak{m}_j ($j = 1, \dots, q$), which are mutually non-equivalent.

Let us consider the G -invariant Riemannian metric on G/H given by

$$(\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{m}_1} + \cdots + x_q \cdot B|_{\mathfrak{m}_q}, \quad x_1, \dots, x_q \in \mathbb{R}_+. \quad (1)$$

Because $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ for any $i \neq j$, any G -invariant metric on G/H is given by (1). Note also that the space of G -invariant symmetric covariant 2-tensors on G/H is given by

$$\{z_1 \cdot B|_{\mathfrak{m}_1} + \cdots + z_q \cdot B|_{\mathfrak{m}_q} \mid z_1, \dots, z_q \in \mathbb{R}\}. \quad (2)$$

In particular, the Ricci tensor r of a G -invariant Riemannian metric on G/H is a G -invariant symmetric covariant 2-tensor on G/H and thus \bar{r} is of the form (2). Let $\{e_\alpha\}$ be a B -orthonormal basis adapted to the decomposition of \mathfrak{g} , i.e., $e_\alpha \in \mathfrak{m}_i$ for some i , and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$). We set $A_{\alpha\beta}^\gamma = B([e_\alpha, e_\beta], e_\gamma)$ so that $[e_\alpha, e_\beta] = \sum_\gamma A_{\alpha\beta}^\gamma e_\gamma$, and set $c_{ij}^k = \left[\begin{smallmatrix} k \\ ij \end{smallmatrix} \right] = \sum_\gamma (A_{\alpha\beta}^\gamma)^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i$, $e_\beta \in \mathfrak{m}_j$, $e_\gamma \in \mathfrak{m}_k$. Then c_{ij}^k is independent of the B -orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and symmetric in all three indices, i.e. $c_{ij}^k = c_{ji}^k = c_{ki}^j$ (see [WZ2]).

Theorem 1. ([PaS]) *Let $d_k = \dim \mathfrak{m}_k$. The components r_1, \dots, r_q of the Ricci tensor r of the metric g of the form (1) on G/H are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \left[\begin{smallmatrix} k \\ ji \end{smallmatrix} \right] - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \left[\begin{smallmatrix} j \\ ki \end{smallmatrix} \right] \quad (k = 1, \dots, q), \quad (3)$$

where the sum is taken over all $i, j = 1, \dots, q$.

2. RIEMANNIAN SUBMERSIONS

Let G be a compact semi-simple Lie group and H, K two closed subgroups of G with $H \subset K$. Then there is a natural fibration $\pi : G/H \rightarrow G/K$ with fiber K/H . Let \mathfrak{p} be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to B , and \mathfrak{q} be the orthogonal complement of \mathfrak{h} in \mathfrak{k} . Then we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q} \oplus \mathfrak{p}$. An $\text{Ad}_G(K)$ -invariant scalar product on \mathfrak{p} defines a G -invariant metric \check{g} on G/K , and an $\text{Ad}_K(H)$ -invariant scalar product on \mathfrak{q} defines an K -invariant metric \hat{g} on K/H . The orthogonal direct sum for these scalar products on $\mathfrak{m} = \mathfrak{q} \oplus \mathfrak{p}$ defines a G -invariant metric g on G/H , called *submersion metric*.

Proposition 1. ([Be, p. 257]) *The map π is a Riemannian submersion from $(G/H, g)$ to $(G/K, \check{g})$ with totally geodesic fibers isometric to $(K/H, \hat{g})$.*

Note that \mathfrak{q} is the vertical subspace of the submersion and \mathfrak{p} is the horizontal subspace. For a Riemannian submersion, O'Neill [ON] has introduced two tensors A and T . Since in our case the fibers are totally geodesic it is $T = 0$. We also have that $A_X Y = \frac{1}{2}[X, Y]_{\mathfrak{q}}$ for any $X, Y \in \mathfrak{p}$. Let now $\{X_i\}$ be an orthonormal basis of \mathfrak{p} and $\{Y_j\}$ be an orthonormal basis of \mathfrak{q} . For $X, Y \in \mathfrak{p}$ we put $g(A_X, A_Y) = \sum_i g(A_X X_i, A_Y X_i)$. Then we have that

$$g(A_X, A_Y) = \frac{1}{4} \sum_i \hat{g}([X, X_i]_{\mathfrak{q}}, [Y, X_i]_{\mathfrak{q}}). \quad (4)$$

Let r, \tilde{r} be the Ricci tensors of the metrics g, \tilde{g} respectively. Then it is easy to see that ([Be, p. 244])

$$r(X, Y) = \tilde{r}(X, Y) - 2g(A_X, A_Y) \quad \text{for } X, Y \in \mathfrak{p}. \quad (5)$$

We remark that there is a corresponding expression $r(U, V)$ for vertical vectors, but it does not contribute additional information in our approach.

Let now $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell$ be a decomposition of \mathfrak{p} into irreducible $\text{Ad}(K)$ -modules and let $\mathfrak{q} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_s$ be a decomposition of \mathfrak{q} into irreducible $\text{Ad}(H)$ -modules. Assume that the $\text{Ad}(K)$ -modules \mathfrak{p}_j ($j = 1, \dots, \ell$) are mutually non equivalent. Note that each irreducible component \mathfrak{p}_j as $\text{Ad}(H)$ -module can be decomposed into irreducible $\text{Ad}(H)$ -modules. To compute the values $\begin{bmatrix} k \\ i, j \end{bmatrix}$ for G/H , we use information from the Riemannian submersion $\pi : (G/H, g) \rightarrow (G/K, \tilde{g})$ with totally geodesic fibers isometric to $(K/H, \hat{g})$. We consider a G -invariant metric on G/H defined by a Riemannian submersion $\pi : (G/H, g) \rightarrow (G/K, \tilde{g})$ given by

$$g = y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell} + z_1 B|_{\mathfrak{q}_1} + \cdots + z_s B|_{\mathfrak{q}_s} \quad (6)$$

for positive real numbers $y_1, \dots, y_\ell, z_1, \dots, z_s$. Then we decompose each irreducible component \mathfrak{p}_j into irreducible $\text{Ad}(H)$ -modules

$$\mathfrak{p}_j = \mathfrak{m}_{j,1} \oplus \cdots \oplus \mathfrak{m}_{j,k_j},$$

where the $\text{Ad}(H)$ -modules $\mathfrak{m}_{j,t}$ ($j = 1, \dots, \ell, t = 1, \dots, k_j$) are mutually non equivalent and are chosen from the irreducible decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ of $\text{Ad}(H)$ -modules. Thus the submersion metric (6) can be written as

$$g = y_1 \sum_{t=1}^{k_1} B|_{\mathfrak{m}_{1,t}} + \cdots + y_\ell \sum_{t=1}^{k_\ell} B|_{\mathfrak{m}_{\ell,t}} + z_1 B|_{\mathfrak{q}_1} + \cdots + z_s B|_{\mathfrak{q}_s}. \quad (7)$$

Note that the metric \tilde{g} on G/K is given by

$$\tilde{g} = y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell} \quad (8)$$

and the metric \hat{g} on K/H are

$$\hat{g} = z_1 B|_{\mathfrak{q}_1} + \cdots + z_s B|_{\mathfrak{q}_s}. \quad (9)$$

Lemma 1. ([ACS]) *Let $d_{j,t} = \dim \mathfrak{m}_{j,t}$. The components $r_{(j,t)}$ ($j = 1, \dots, \ell, t = 1, \dots, k_j$) of the Ricci tensor r for the metric (7) on G/H are given by*

$$r_{(j,t)} = \tilde{r}_j - \frac{1}{2d_{j,t}} \sum_{i=1}^s \sum_{j',t'} \frac{z_i}{y_j y_{j'}} \begin{bmatrix} i \\ (j,t) \quad (j',t') \end{bmatrix}, \quad (10)$$

where \tilde{r}_j are the components of Ricci tensor \tilde{r} for the metric \tilde{g} on G/K .

Notice that when metric (6) is viewed as a metric (1) then the horizontal part of $r_{(j,t)}$ equals to \tilde{r}_j ($j = 1, \dots, \ell$), in particular, it is independent of t .

3. DECOMPOSITION ASSOCIATED TO GENERALIZED FLAG MANIFOLDS

In this section we review briefly the Lie theoretic description of a flag manifold in terms of painted Dynkin diagrams, and next we recall some notions from the geometry and the topology of such a space.

Let G be a compact semi-simple Lie group, \mathfrak{g} the Lie algebra of G and \mathfrak{t} a maximal abelian subalgebra of \mathfrak{g} . We denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ the complexification of \mathfrak{g} and \mathfrak{t} , respectively. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We assume that $\dim_{\mathbb{C}} \mathfrak{t}^{\mathbb{C}} = \ell = \text{rk } \mathfrak{g}^{\mathbb{C}}$. We identify an element of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$ with an element of $\sqrt{-1}\mathfrak{t}$, by the duality defined by the Killing form of $\mathfrak{g}^{\mathbb{C}}$. Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}}$. Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a fundamental system of Δ and $\{\Lambda_1, \dots, \Lambda_\ell\}$ the fundamental weights of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π , that is $2(\Lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$, for any $1 \leq i, j \leq \ell$. We choose a subset $\Pi_0 \subset \Pi$ and we set $\Pi_M = \Pi \setminus \Pi_0 = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ ($1 \leq \alpha_{i_1} < \dots < \alpha_{i_r} \leq \ell$).

We put $[\Pi_0] = \Delta \cap \text{span}_{\mathbb{Z}}\{\Pi_0\}$ and $[\Pi_0]^+ = \Delta^+ \cap \text{span}_{\mathbb{Z}}\{\Pi_0\}$, where $\text{span}_{\mathbb{Z}}\{\Pi_0\}$ denotes the subspace of $\sqrt{-1}\mathfrak{t}$ generated by Π_0 with integer coefficients, and Δ^+ is the set of all positive roots relative to Π . Take a Weyl basis $\{E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}} : \alpha \in \Delta\}$, and set $A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$. Then the Lie algebra \mathfrak{g} , is a real form of $\mathfrak{g}^{\mathbb{C}}$ which can be identified with the fixed-point set \mathfrak{g}^τ of the complex conjugation τ in $\mathfrak{g}^{\mathbb{C}}$, that means $\mathfrak{g}^\tau = \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}$ (see [AC2]). Moreover, the subalgebra $\mathfrak{u} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in [\Pi_0] \cup \Delta^+} \mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a *parabolic subalgebra* of $\mathfrak{g}^{\mathbb{C}}$ since it contains the *Borel subalgebra* $\mathfrak{b} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$.

Let $G^{\mathbb{C}}$ be a simply connected complex semi-simple Lie group whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$ and U the parabolic subgroup of $G^{\mathbb{C}}$ generated by \mathfrak{u} . Since U is connected, the complex homogeneous manifold $G^{\mathbb{C}}/U$ is simply connected (and compact). In fact G acts transitively on $G^{\mathbb{C}}/U$ with isotropy group the connected closed subgroup $H = G \cap U \subset G$, thus $G^{\mathbb{C}}/U = G/H$ as C^∞ -manifolds. This identification implies that $G^{\mathbb{C}}/U$ carries a G -invariant Kähler metric. Notice that $H = G \cap U$ is the centralizer of a torus $S \subset T$ in G , where T is the maximal torus generated from the ad-diagonal subalgebra \mathfrak{t} . Thus $\text{rk } G = \text{rk } H$. The homogeneous space $M = G^{\mathbb{C}}/U = G/H$ is called *generalized flag manifold*, and any generalized flag manifold is constructed in this way. Let \mathfrak{h} be the Lie algebra of H and let $\mathfrak{h}^{\mathbb{C}}$ be its complexification. Due to the inclusion $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{h}^{\mathbb{C}} \subset \mathfrak{u}$ we obtain a direct sum decomposition $\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{n}$, such that $\mathfrak{g} \cap \mathfrak{u} = \mathfrak{h}$, where the nilradical \mathfrak{n} of \mathfrak{u} and the subalgebra $\mathfrak{h}^{\mathbb{C}}$ are given by $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+ - [\Pi_0]^+} \mathfrak{g}_\alpha^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in [\Pi_0]} \mathfrak{g}_\alpha^{\mathbb{C}}$, respectively. Consequently the real subalgebra \mathfrak{h} has the form by $\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in [\Pi_0]^+} \{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}$.

From now on we will denote by $\tilde{\alpha} = \sum_{k=1}^\ell c_k \alpha_k$ the highest (or maximal) root of $\mathfrak{g}^{\mathbb{C}}$, it means that $c_k \geq m_k$ for any other positive root $\alpha = \sum_{k=1}^\ell m_k \alpha_k \in \Delta^+$. Next we will call *height* of a simple root $\alpha_i \in \Pi$ the positive integer c_i and we will use the map $\text{ht} : \Pi \rightarrow \mathbb{Z}_+, \alpha_i \mapsto \text{ht}(\alpha_i) := c_i$.

Proposition 2. ([BuR, Proposition 4.3]) *Let \mathfrak{z} be the center of the nilpotent Lie algebra \mathfrak{n} . Then we have $\text{ad}(\mathfrak{h}^{\mathbb{C}})(\mathfrak{z}) \subset \mathfrak{z}$ and the action of $\mathfrak{h}^{\mathbb{C}}$ on \mathfrak{z} is irreducible. Moreover, the $\text{ad}(\mathfrak{h}^{\mathbb{C}})$ -module \mathfrak{z} is generated by the highest root space $\mathfrak{g}_{\tilde{\alpha}}^{\mathbb{C}}$.*

We denote by \mathfrak{h}_0 the center of \mathfrak{h} , and $\mathfrak{h}_0^{\mathbb{C}}$ its complexification. Since $\mathfrak{h}^{\mathbb{C}}$ is a reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$, it admits the decomposition $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}_0^{\mathbb{C}} \oplus \mathfrak{h}_{ss}^{\mathbb{C}}$, where $\mathfrak{h}_{ss}^{\mathbb{C}}$ is the semi-simple part of $\mathfrak{h}^{\mathbb{C}}$, given by $\mathfrak{h}_{ss}^{\mathbb{C}} = [\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] = \bigoplus_{\alpha \in \Pi_0} \mathbb{C}\alpha \oplus \bigoplus_{\alpha \in [\Pi_0]} \mathfrak{g}_\alpha^{\mathbb{C}}$. The set $[\Pi_0]$ is the root system of $\mathfrak{h}_{ss}^{\mathbb{C}}$ and Π_0 is a basis of simple roots for it. For convenience, we will denote the set $[\Pi_0]$ by Δ_H . We set $\Delta_M = \Delta \setminus \Delta_H$. Roots belong to Δ_M are called *complementary roots* and they have a significant role in the geometry of $M = G/H$. For example, let \mathfrak{m} be the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to B . Then we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and we identify \mathfrak{m} with the tangent space $T_o G/H$ in $o = eH \in G/H$. Set $\Delta_M^+ = \Delta^+ \setminus \Delta_H^+$, where Δ_H^+ is the system of positive roots of $\mathfrak{h}^{\mathbb{C}}$, i.e., $\Delta_H^+ = [\Pi_0]^+$. Then

$$\mathfrak{m} = \bigoplus_{\alpha \in \Delta_M^+} \{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}. \quad (11)$$

The complexified tangent space $\mathfrak{m}^{\mathbb{C}}$ is given by $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Delta_M} \mathfrak{g}_\alpha^{\mathbb{C}}$, and the set $\{E_\alpha : \alpha \in \Delta_M\}$ is a basis of $\mathfrak{m}^{\mathbb{C}}$. Note that although the set Π_M consists of all these complementary roots which are simple, is not in general a basis of Δ_M , that is Δ_M is not in general a root system.

Generalized flag manifolds $M = G/H$ of a compact connected simple Lie group G are classified by using the Dynkin diagram of G , as follows: Let $\Gamma = \Gamma(\Pi)$ be the Dynkin diagram corresponding to the base of simple roots Π of the root system Δ of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$.

Definition 1. *The painted Dynkin diagram of $M = G/H$ is obtained from the Dynkin diagram $\Gamma = \Gamma(\Pi)$ by painting black the nodes which correspond to the simple roots of Π_M . The sub-diagram of white nodes with the connecting lines between them determines the semi-simple part \mathfrak{h}_{ss} of the Lie algebra \mathfrak{h} of H , and each black node gives rise to one $\mathfrak{u}(1)$ -summand (their totality forms the center \mathfrak{h}_0 of \mathfrak{h}).*

Thus the painted Dynkin diagram determines the isotropy group H and the space $M = G/H$ completely. It should be noted that the resulting painted Dynkin diagram does not depend on the choice of a maximal abelian subalgebra \mathfrak{t} and hence of Δ . On the other hand the necessity of making a choice of a base Π for Δ (or equivalently of an ordering Δ^+ in Δ) reduces the number of painted Dynkin diagrams. By using certain rules to determine whether different painted Dynkin diagrams define isomorphic flag manifolds, one can obtain all flag manifolds G/H of a compact connected simple Lie group G (cf. [AA]).

Remark 1. The (real) dimension of the center \mathfrak{h}_0 of the subalgebra \mathfrak{h} is equal to the number of black nodes in the painted Dynkin diagram of $M = G/H$, or equivalent equal to the number of $\mathfrak{u}(1)$ summands in the decomposition of \mathfrak{h} . By assuming that $\Pi_M = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, it follows that the fundamental weights $\Lambda_{i_1}, \dots, \Lambda_{i_r}$ form a basis of the dual space \mathfrak{h}_0^* of \mathfrak{h}_0 . Since $\mathfrak{h}_0^* \cong \mathfrak{h}_0$ via the Killing form of \mathfrak{g} , we obtain

$\dim \mathfrak{h}_0 = r = |\Pi_M|$ where $|\Pi_M|$ is the cardinality of Π_M (cf. [APe]). From [BHi, p. 507] it is well-known that $H^2(M; \mathbb{R}) = H^1(H; \mathbb{R}) = \mathfrak{h}_0$. Thus the second Betti number $b_2(M)$ of the flag manifold $M = G/H$ is equal to $\dim \mathfrak{h}_0$ and it is obtained directly from the painted Dynkin diagram. Moreover, any flag manifold $M = G/H$ of a simple Lie group G with $b_2(M) = r$, is determined by a subset $\Pi_M \subset \Pi$ with $|\Pi_M| = r$ and it is constructed in the above way.

From now on we assume that G is *simple* and we choose a subset $\Pi_0 \subset \Pi$ such that $\Pi_M = \Pi - \Pi_0 = \{\alpha_i\}$, for some fixed i with $1 \leq i \leq \ell$. Then the corresponding flag manifold $M = G/H$ is such that $\dim \mathfrak{h}_0 = 1$ and $b_2(M) = 1$. We also assume that $\text{ht}(\alpha_i) = N \in \mathbb{Z}^+$. To an integer k with $1 \leq k \leq N$ we associate the set $\Delta^+(\alpha_i, k) = \left\{ \alpha \in \Delta^+ \mid \alpha = \sum_{j=1}^{\ell} m_j \alpha_j, m_i = k \right\}$. Then it is obvious that $\Delta_M^+ = \Delta^+ \setminus \Delta_H^+ = \bigcup_{1 \leq k \leq N} \Delta^+(\alpha_i, k)$. We define a subspace \mathfrak{n}_k of the nilradical \mathfrak{n} by $\mathfrak{n}_k = \bigoplus_{\alpha \in \Delta^+(\alpha_i, k)} \mathbb{C} E_\alpha$. Then \mathfrak{n}_k ($k = 1, \dots, N$) are $\text{ad}(\mathfrak{h}^{\mathbb{C}})$ -invariant subspaces, and $\mathfrak{n} = \bigoplus_{j=1}^N \mathfrak{n}_j$ is an irreducible decomposition of \mathfrak{n} (see [Wo2]). In view of Proposition 2 we have that $\mathfrak{z} = \mathfrak{n}_N$. We also define subspaces \mathfrak{m}_k of \mathfrak{m} , given by

$$\mathfrak{m}_k = \bigoplus_{\alpha \in \Delta^+(\alpha_i, k)} \{ \mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}\sqrt{-1}(E_\alpha - E_{-\alpha}) \}. \quad (12)$$

Note that \mathfrak{m}_k are $\text{Ad}(H)$ -invariant submodules of \mathfrak{m} which are mutually inequivalent each other, for any $k = 1, \dots, N$ ([Kim]). We also recall the following useful inclusions (see for example [AC2]):

$$[\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i, \quad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h} + \mathfrak{m}_{2i}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{|i-j|} \quad (i \neq j). \quad (13)$$

By using (11), we get a characterization of \mathfrak{m} in terms of the submodules \mathfrak{m}_k :

Lemma 2. *Let $M = G^{\mathbb{C}}/U = C/H$ be a flag manifold of a compact connected simple Lie group G , defined by a subset $\Pi_M = \{\alpha_i : \text{ht}(\alpha_i) = N\} \subset \Pi$. Then, $\mathfrak{m} = T_o M$ admits a decomposition $\mathfrak{m} = \bigoplus_{k=1}^N \mathfrak{m}_k$ into N irreducible, inequivalent $\text{Ad}(H)$ -submodules \mathfrak{m}_k defined by (12). Moreover, it is $d_k = \dim_{\mathbb{R}} \mathfrak{m}_k = 2 \cdot |\Delta^+(\alpha_i, k)|$, for any $1 \leq k \leq N$.*

Note that according to the notation of §1, for the space $M = G^{\mathbb{C}}/U = G/H$ in Lemma 2, it is $N = q$.

Remark 2. It is well known (cf. [APe], [Tak], [AC3]) that for a flag manifold G/H , there is a 1-1 correspondence between G -invariant complex structures J and compatible G -invariant Kähler-Einstein metrics h_J , given by $J \leftrightarrow h_J = \{h_\alpha = (\delta_{\mathfrak{m}}, \alpha) : \alpha \in \Delta_M^+\}$, where $h_\alpha = h_J(E_\alpha, E_{-\alpha})$ are the components of the metric h_J with respect to the base $\{E_\alpha : \alpha \in \Delta_M\}$ of $\mathfrak{m}^{\mathbb{C}}$. The weight $\delta_{\mathfrak{m}} = (1/2) \sum_{\beta \in \Delta_M^+} \beta \in \sqrt{-1}\mathfrak{h}_0$ is called *Koszul form*. If we assume that M is defined by a subset $\Pi_M = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, then the following relation holds: $2\delta_{\mathfrak{m}} = u_{i_1} \cdot \Lambda_{\alpha_{i_1}} + \dots + u_{i_r} \cdot \Lambda_{\alpha_{i_r}}$. The positive integers $u_{i_1} > 0, \dots, u_{i_r} > 0$ are called *Koszul numbers*.

Proposition 3. ([BHi]) *Let $M = G^{\mathbb{C}}/U = G/H$ be a flag manifold defined as in Lemma 2. Then M admits a unique G -invariant Kähler-Einstein metric given by*

$$h_J = B|_{\mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2} + \dots + N \cdot B|_{\mathfrak{m}_N}. \quad (14)$$

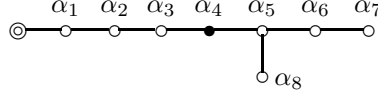
Proof. We give a short proof here since one is difficult to find it in the literature. By [BHi, Proposition 13.8] we know that M admits a unique G -invariant complex structure J induced by the invariant ordering $\Delta_M^+ = \Delta^+ / \Delta_H^+$ (we identify J with its conjugate \bar{J} which is induced by the invariant ordering $\Delta_M^- = -\Delta_M^+$). The complex structure J is described by an $\text{ad}(\mathfrak{h}^{\mathbb{C}})$ -invariant endomorphism J_o on $\mathfrak{m}^{\mathbb{C}}$ with $J_o^2 = -\text{Id}_{\mathfrak{m}^{\mathbb{C}}}$, explicitly determined by the formulae $J_o E_{\pm\alpha} = \pm\sqrt{-1}E_{\pm\alpha}$, for any $\alpha \in \Delta_M^+$. In view of Remark 2, M admits a unique G -invariant Kähler-Einstein metric h_J compatible with J . Because $\Pi_M = \{\alpha_i : \text{ht}(\alpha_i) = N\}$, (where i is fixed, $1 \leq i \leq \ell$), we have $\delta_{\mathfrak{m}} = u_i/2 \cdot \Lambda_i$ with $u_i > 0$. From Lemma 2 it is $\mathfrak{m} = \bigoplus_{k=1}^N \mathfrak{m}_k$, thus the G -invariant metric h_J on M has the form $h_J = \sum_{k=1}^N h_k \cdot B|_{\mathfrak{m}_k}$ with $(h_1, \dots, h_N) \in \mathbb{R}_+^N$. Here by h_k we denote the component of the metric h_J on the specific submodule \mathfrak{m}_k , for any $1 \leq k \leq N$, i.e. $h_k = h_J(E_\alpha, E_{-\alpha})$ with $\alpha \in \Delta^+(\alpha_i, k)$; by Remark 2 it is defined as follows: $h_k = h_J(E_\alpha, E_{-\alpha}) = (\delta_{\mathfrak{m}}, \alpha)$ with $\alpha \in \Delta^+(\alpha_i, k)$. Because $(\Lambda_i, \alpha_i) = (\alpha_i, \alpha_i)/2$ it is easy to see that

$$h_k = (\delta_{\mathfrak{m}}, \alpha) = \left(\frac{u_i}{2} \cdot \Lambda_i, m_1 \alpha_1 + \dots + k \alpha_i + \dots + m_\ell \alpha_\ell\right) = \left(\frac{u_i}{2} \cdot \Lambda_i, k \alpha_i\right) = k \cdot u_i \cdot (\alpha_i, \alpha_i).$$

Since the simple root α_i is fixed, the number $u_i \cdot (\alpha_i, \alpha_i)$ is constant and independent of the integer k for any $1 \leq k \leq N$. By normalizing the metric the proof is complete. \square

4. HOMOGENEOUS EINSTEIN METRICS ON $E_8 / U(1) \times SU(4) \times SU(5)$

4.1. The construction of the homogeneous Einstein equation on $E_8 / U(1) \times SU(4) \times SU(5)$. Let $G = E_8$. A basis of simple roots for the root system of E_8 is given by $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_7 = e_7 - e_8, \alpha_8 = e_6 + e_7 + e_8\}$, and $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ (cf. [AC3]). We set $\Pi_M = \{\alpha_4\}$, thus $\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. So we obtain the (extended) painted Dynkin diagram (the double circle denotes the negative of $\tilde{\alpha}$)



It defines the flag manifold $M = G/H = E_8 / U(1) \times SU(4) \times SU(5)$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition of \mathfrak{g} with respect to B . Because $\text{ht}(\alpha_4) = 5$, from Lemma 2 it follows that $N = 5 = q$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$. We consider an E_8 -invariant Riemannian metric (\cdot, \cdot) on $G/H = E_8 / U(1) \times SU(4) \times SU(5)$ given by

$$(\cdot, \cdot) = x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5}, \quad (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}_+^5. \quad (15)$$

By applying Theorem 1, we obtain that:

Proposition 4. *The components r_i of the Ricci tensor r for the G -invariant metric (\cdot, \cdot) on G/H defined by (15), are given as follows*

$$\left\{ \begin{array}{l} r_1 = \frac{1}{2x_1} - \frac{1}{2d_1} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \frac{x_2}{x_1^2} + \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 23 \end{smallmatrix} \right] \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right) \\ \quad + \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 34 \end{smallmatrix} \right] \left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3} \right) + \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 45 \end{smallmatrix} \right] \left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4} \right) \\ r_2 = \frac{1}{2x_2} + \frac{1}{4d_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) - \frac{1}{2d_2} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \frac{x_4}{x_2^2} \\ \quad + \frac{1}{2d_2} \left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix} \right] \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1} \right) + \frac{1}{2d_2} \left[\begin{smallmatrix} 2 \\ 35 \end{smallmatrix} \right] \left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3} \right) \\ r_3 = \frac{1}{2x_3} + \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] \left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2} \right) + \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 14 \end{smallmatrix} \right] \left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} \right) \\ \quad + \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 25 \end{smallmatrix} \right] \left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2} \right) \\ r_4 = \frac{1}{2x_4} + \frac{1}{4d_4} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \left(\frac{x_4}{x_2^2} - \frac{2}{x_4} \right) + \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] \left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1} \right) \\ \quad + \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 15 \end{smallmatrix} \right] \left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4} \right) \\ r_5 = \frac{1}{2x_5} + \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} \right) + \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] \left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} \right). \end{array} \right. \quad (16)$$

From Proposition 3, we know that the metric $B|_{\mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2} + 3 \cdot B|_{\mathfrak{m}_3} + 4 \cdot B|_{\mathfrak{m}_4} + 5 \cdot B|_{\mathfrak{m}_5}$ is the unique Kähler-Einstein on G/H . By substituting these values in the system $\{r_1 = r_2 = r_3 = r_4 = r_5\}$, we obtain

$$\begin{aligned} & \frac{1}{2} - \frac{1}{d_1} \left(\left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] \right) = \frac{1}{4} + \frac{1}{d_2} \left(\frac{1}{4} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] - \frac{1}{2} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] - \frac{1}{2} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] - \frac{1}{2} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \right) \\ &= \frac{1}{6} + \frac{1}{d_3} \left(\frac{1}{3} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] - \frac{1}{3} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] - \frac{1}{3} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \right) = \frac{1}{8} + \frac{1}{d_4} \left(\frac{1}{4} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] - \frac{1}{4} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] + \frac{1}{8} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \right) \\ &= \frac{1}{10} + \frac{1}{d_5} \left(\frac{1}{5} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] + \frac{1}{5} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \right). \end{aligned} \quad (17)$$

4.2. Use of submersion. From (17) we obtain a system with four equations and six unknowns, namely the triples $\left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right], \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right], \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right], \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right], \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right]$, and $\left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right]$. In order to compute them explicitly, we make use of Lemma

1. We put $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_5$, $\mathfrak{k}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_5$, $\mathfrak{p}_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_4$, $\mathfrak{p}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_3$ and $\mathfrak{q}_1 = \mathfrak{m}_5$. Then \mathfrak{k} is a subalgebra of \mathfrak{g} . By using (13) we get that

$$[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{p}_2 \oplus \mathfrak{k}, \quad [\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2, \quad [\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{p}_1 \oplus \mathfrak{k}. \quad (18)$$

Thus, we obtain an irreducible decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2$ as $\text{Ad}(K)$ -modules, which are mutually non-equivalent (cf. [WZ1, p. 575]).

Note that we have an irreducible decomposition

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \quad (19)$$

as $\text{Ad}(H)$ -modules, where \mathfrak{h}_0 is the center of \mathfrak{h} and $\mathfrak{h}_1 = \mathfrak{su}(4)$, $\mathfrak{h}_2 = \mathfrak{su}(5)$, and that $d_0 = \dim \mathfrak{h}_0 = 1$, $d_1 = \dim \mathfrak{h}_1 = 15$ and $d_2 = \dim \mathfrak{h}_2 = 24$. Also, by applying the second part of Lemma 2 we obtain that $d_1 = \dim \mathfrak{m}_1 = 80$, $d_2 = \dim \mathfrak{m}_2 = 60$, $d_3 = \dim \mathfrak{m}_3 = 40$, $d_4 = \dim \mathfrak{m}_4 = 20$ and $d_5 = \dim \mathfrak{m}_5 = 8$.

Proposition 5. *In the decomposition (19) we can take the ideal \mathfrak{h}_2 such that $[\mathfrak{h}_2, \mathfrak{m}_5] = \{0\}$.*

Proof. We can assume that $\mathfrak{h}_2 \neq \{0\}$. Note that there is only a simple root $\alpha_{j_0} = \alpha_8$ with $(\alpha_{j_0}, \tilde{\alpha}) \neq 0$ and thus we can take the ideal \mathfrak{h}_2 so that $[\mathfrak{h}_2^{\mathbb{C}}, E_{\tilde{\alpha}}] = \{0\}$. Since $\mathfrak{n}_5 = [\mathfrak{h}^{\mathbb{C}}, E_{\tilde{\alpha}}]$, we have that $[\mathfrak{h}_2^{\mathbb{C}}, \mathfrak{n}_5] = [\mathfrak{h}_2^{\mathbb{C}}, [\mathfrak{h}^{\mathbb{C}}, E_{\tilde{\alpha}}]] \subset [[\mathfrak{h}_2^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}], E_{\tilde{\alpha}}] + [\mathfrak{h}^{\mathbb{C}}, [\mathfrak{h}_2^{\mathbb{C}}, E_{\tilde{\alpha}}]] = \{0\}$. By the definition of \mathfrak{m}_5 , we get the result. \square

From Proposition 5, we see that \mathfrak{k}_1 is also a subalgebra of \mathfrak{g} . In particular it is $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_2 = \mathfrak{su}(5)$, and for dimensional reasons we also obtain that $\mathfrak{k}_1 = \mathfrak{su}(5)$.

Since $\mathfrak{h} \subset \mathfrak{k}$, we determine a fibration $G/H \rightarrow G/K$, given by $E_8 / \text{U}(1) \times \text{SU}(4) \times \text{SU}(5) \rightarrow E_8 / \text{SU}(5) \times \text{SU}(5)$. The base space $G/K = E_8 / \text{SU}(5) \times \text{SU}(5)$ has two isotropy summands, namely \mathfrak{p}_1 and \mathfrak{p}_2 . We consider a Riemannian submersion $\pi : (G/H, g) \rightarrow (G/K, \check{g})$ with totally geodesic fibers isometric to $(K/H, \hat{g})$.

Note that a G -invariant metric \check{g} on $G/K = E_8 / \text{SU}(5) \times \text{SU}(5)$ is given by

$$\check{g} = y_1 \cdot B|_{\mathfrak{p}_1} + y_2 \cdot B|_{\mathfrak{p}_2}, \quad (y_1, y_2) \in \mathbb{R}_+^2, \quad (20)$$

a G -invariant metric g on $G/H = E_8 / \text{U}(1) \times \text{SU}(4) \times \text{SU}(5)$ is given by

$$g = y_1 \cdot B|_{\mathfrak{p}_1} + y_2 \cdot B|_{\mathfrak{p}_2} + z_1 \cdot B|_{\mathfrak{q}_1}, \quad (y_1, y_2, z_1) \in \mathbb{R}_+^3 \quad (21)$$

and a K -invariant metric \hat{g} on $K/H \simeq \text{SU}(5) / \text{U}(1) \times \text{SU}(4)$ is given by

$$\hat{g} = z_1 \cdot B|_{\mathfrak{q}_1}, \quad z_1 \in \mathbb{R}_+. \quad (22)$$

Notice that the metric (21) can be written as the metric of the form (15):

$$g = y_1 \cdot B|_{\mathfrak{m}_1} + y_2 \cdot B|_{\mathfrak{m}_2} + y_2 \cdot B|_{\mathfrak{m}_3} + y_1 \cdot B|_{\mathfrak{m}_4} + z_1 \cdot B|_{\mathfrak{m}_5}. \quad (23)$$

From (16) we obtain components r_i of the Ricci tensor r for the metric (23) on G/H as follows:

$$\left\{ \begin{array}{l} r_1 = \frac{1}{2y_1} - \frac{1}{2d_1} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \frac{y_2}{y_1^2} + \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 23 \end{smallmatrix} \right] \left(\frac{y_1}{y_2^2} - \frac{2}{y_1} \right) - \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 34 \end{smallmatrix} \right] \frac{y_2}{y_1^2} - \frac{1}{2d_1} \left[\begin{smallmatrix} 1 \\ 45 \end{smallmatrix} \right] \frac{z_1}{y_1^2} \\ r_2 = \frac{1}{2y_2} + \frac{1}{4d_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) - \frac{1}{2d_2} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \frac{y_1}{y_2^2} - \frac{1}{2d_2} \left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix} \right] \frac{y_1}{y_2^2} - \frac{1}{2d_2} \left[\begin{smallmatrix} 2 \\ 35 \end{smallmatrix} \right] \frac{z_1}{y_2^2} \\ r_3 = \frac{1}{2y_2} - \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] \frac{y_1}{y_2^2} + \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 14 \end{smallmatrix} \right] \left(\frac{y_2}{y_1 y_2} - \frac{2}{y_2} \right) - \frac{1}{2d_3} \left[\begin{smallmatrix} 3 \\ 25 \end{smallmatrix} \right] \frac{z_1}{y_2^2} \\ r_4 = \frac{1}{2y_1} + \frac{1}{4d_4} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \left(\frac{y_1}{y_2^2} - \frac{2}{y_1} \right) - \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] \frac{y_2}{y_1^2} - \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 15 \end{smallmatrix} \right] \frac{z_1}{y_1^2} \\ r_5 = \frac{1}{2z_1} + \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \left(\frac{z_1}{y_2^2} - \frac{2}{z_1} \right) + \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] \left(\frac{z_1}{y_1^2} - \frac{2}{z_1} \right). \end{array} \right. \quad (24)$$

Now we put that $\check{d}_1 = \dim \mathfrak{p}_1 = 100$ and $\check{d}_2 = \dim \mathfrak{p}_2 = 100$. Note that the components \check{r}_i of the Ricci tensor \check{r} of the E_8 -invariant metric \check{g} on E_8 / K defined by (20), are given as follows:

$$\left\{ \begin{array}{l} \check{r}_1 = \frac{1}{2y_1} + \frac{y_1}{4\check{d}_1 y_2^2} \left[\begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right] - \frac{1}{2\check{d}_1} \left(\frac{y_2}{y_1^2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] + \frac{1}{y_1} \left[\begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right] \right) \\ \check{r}_2 = \frac{1}{2y_2} + \frac{y_2}{4\check{d}_2 y_1^2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] - \frac{1}{2\check{d}_2} \left(\frac{y_1}{y_2^2} \left[\begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right] + \frac{1}{y_2} \left[\begin{smallmatrix} 1 \\ 12 \end{smallmatrix} \right] \right). \end{array} \right. \quad (25)$$

From Lemma 1, by taking the horizontal part of r_1 and r_4 , and r_2 and r_3 , we see that

$$\left\{ \begin{array}{l} \check{r}_1 = \frac{1}{2y_1} - \frac{1}{2d_1} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \frac{y_2}{y_1^2} + \frac{1}{2d_1} \begin{bmatrix} 1 \\ 23 \end{bmatrix} \left(\frac{y_1}{y_2^2} - \frac{2}{y_1} \right) - \frac{1}{2d_1} \begin{bmatrix} 1 \\ 34 \end{bmatrix} \frac{y_2}{y_1^2} \\ \quad = \frac{1}{2y_1} + \frac{1}{4d_4} \begin{bmatrix} 4 \\ 22 \end{bmatrix} \left(\frac{y_1}{y_2^2} - \frac{2}{y_1} \right) - \frac{1}{2d_4} \begin{bmatrix} 4 \\ 13 \end{bmatrix} \frac{y_2}{y_1^2} \\ \check{r}_2 = \frac{1}{2y_2} + \frac{1}{4d_2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) - \frac{1}{2d_2} \begin{bmatrix} 4 \\ 22 \end{bmatrix} \frac{y_1}{y_2^2} - \frac{1}{2d_2} \begin{bmatrix} 2 \\ 13 \end{bmatrix} \frac{y_1}{y_2^2} \\ \quad = \frac{1}{2y_2} - \frac{1}{2d_3} \begin{bmatrix} 3 \\ 12 \end{bmatrix} \frac{y_1}{y_2^2} + \frac{1}{2d_3} \begin{bmatrix} 3 \\ 14 \end{bmatrix} \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right). \end{array} \right. \quad (26)$$

Hence we conclude that the following equalities must hold:

$$\left\{ \begin{array}{l} \frac{1}{2\check{d}_1} \begin{bmatrix} 2 \\ 12 \end{bmatrix} = \frac{1}{d_1} \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \frac{1}{2d_4} \begin{bmatrix} 4 \\ 22 \end{bmatrix}, \quad \frac{1}{2\check{d}_1} \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \frac{1}{2d_1} \begin{bmatrix} 2 \\ 11 \end{bmatrix} + \frac{1}{2d_1} \begin{bmatrix} 4 \\ 13 \end{bmatrix} = \frac{1}{2d_4} \begin{bmatrix} 4 \\ 13 \end{bmatrix} \\ \frac{1}{2\check{d}_2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \frac{1}{2d_2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \frac{1}{d_3} \begin{bmatrix} 4 \\ 13 \end{bmatrix}, \quad \frac{1}{2\check{d}_2} \begin{bmatrix} 1 \\ 22 \end{bmatrix} = \frac{1}{d_2} \begin{bmatrix} 4 \\ 22 \end{bmatrix} + \frac{1}{d_2} \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \frac{1}{d_3} \begin{bmatrix} 3 \\ 12 \end{bmatrix} \end{array} \right\}. \quad (27)$$

From equations (17) and (27), we get a system of equations:

$$\left. \begin{array}{l} 60 - 4 \begin{bmatrix} 2 \\ 11 \end{bmatrix} - \begin{bmatrix} 3 \\ 12 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 13 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 14 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 22 \end{bmatrix} + \begin{bmatrix} 5 \\ 23 \end{bmatrix} = 0 \\ 20 + \begin{bmatrix} 2 \\ 11 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 12 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 13 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 22 \end{bmatrix} = 0 \\ 20 + 4 \begin{bmatrix} 2 \\ 11 \end{bmatrix} - 10 \begin{bmatrix} 4 \\ 13 \end{bmatrix} + 6 \begin{bmatrix} 5 \\ 14 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 22 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 23 \end{bmatrix} = 0 \end{array} \right\} \begin{array}{l} 4 + 2 \begin{bmatrix} 4 \\ 13 \end{bmatrix} - 6 \begin{bmatrix} 5 \\ 14 \end{bmatrix} + \begin{bmatrix} 4 \\ 22 \end{bmatrix} - 4 \begin{bmatrix} 5 \\ 23 \end{bmatrix} = 0 \\ \begin{bmatrix} 2 \\ 11 \end{bmatrix} - 3 \begin{bmatrix} 4 \\ 13 \end{bmatrix} = 0 \\ \begin{bmatrix} 3 \\ 12 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 22 \end{bmatrix} = 0. \end{array} \quad (28)$$

By solving system (28) we obtain explicitly the values of all non-zero triples of G/H .

Proposition 6. *For the G -invariant metric (\cdot, \cdot) on $M = G/H = E_8/U(1) \times SU(4) \times SU(5)$, the non-zero structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ are given by $\begin{bmatrix} 2 \\ 11 \end{bmatrix} = 12$, $\begin{bmatrix} 3 \\ 12 \end{bmatrix} = 8$, $\begin{bmatrix} 4 \\ 13 \end{bmatrix} = 4$, $\begin{bmatrix} 5 \\ 14 \end{bmatrix} = 4/3$, $\begin{bmatrix} 4 \\ 22 \end{bmatrix} = 4$, and $\begin{bmatrix} 5 \\ 23 \end{bmatrix} = 2$.*

4.2.1. Solutions of the homogeneous Einstein equation. It is obvious that due to Proposition 6, the components r_i ($1 \leq i \leq 5$) of the Ricci tensor are completely determined by equation (16). Thus, a G -invariant metric on G/H given by (15), is an Einstein metric, if and only if it is a positive real solution of the system of equations $\{r_1 - r_2 = 0, r_2 - r_3 = 0, r_3 - r_4 = 0, r_4 - r_5 = 0\}$. We normalize our equations by setting $x_1 = 1$. Then, we obtain the following system of polynomial equations:

$$\left\{ \begin{array}{l} f_1 = -15x_2^3x_3x_4x_5 - 14x_2^3x_4x_5 - 2x_2^3x_4 - 3x_2^2x_3^2x_5 - x_2^2x_3x_4^2 + 60x_2^2x_3x_4x_5 \\ \quad + x_2^2x_3 - 3x_2^2x_4^2x_5 + 3x_2^2x_5 + 2x_2x_3^2x_4x_5 + 2x_2x_3^2x_4 - x_2x_5^2(x_2x_3 - 2x_4) \\ \quad - 48x_2x_3x_4x_5 + 14x_2x_4x_5 + 4x_3x_4^2x_5 = 0, \\ f_2 = 6x_2^3x_3x_4x_5 + 20x_2^3x_4x_5 + 5x_2^3x_4 - 6x_2^2x_3^2x_5 + 6x_2^2x_4^2x_5 - 60x_2^2x_4x_5 + 6x_2^2x_5 \\ \quad - 20x_2x_3^2x_4x_5 - 5x_2x_3^2x_4 + 48x_2x_3x_4x_5 + x_2x_4x_5^2 + 4x_2x_4x_5 - 4x_3x_4^2x_5 = 0, \\ f_3 = -12x_2^3x_4x_5 - 3x_2^3x_4 + 18x_2^2x_3^2x_5 - 4x_2^2x_3x_4^2 - 48x_2^2x_3x_5 + 4x_2^2x_3 \\ \quad - 18x_2^2x_4^2x_5 + 60x_2^2x_4x_5 + 6x_2^2x_5 + 12x_2x_3^2x_4x_5 + 3x_2x_3^2x_4 + x_2x_5^2(4x_2x_3 - 3x_4) \\ \quad - 12x_2x_4x_5 - 6x_3x_4^2x_5 = 0, \\ f_4 = 15x_2^3x_4 - 12x_2^2x_3^2x_5 + 14x_2^2x_3x_4^2 - 60x_2^2x_3x_4 + 48x_2^2x_3x_5 + 6x_2^2x_3 + 12x_2^2x_4^2x_5 \\ \quad - 12x_2^2x_5 + 15x_2x_3^2x_4 - x_2x_5^2(14x_2x_3 + 15x_4) + 6x_3x_4^2x_5 = 0 \end{array} \right. \quad (29)$$

To find non-zero solutions of equations (29), we consider a polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ and an ideal I generated by $\{f_1, f_2, f_3, f_4, yx_2x_3x_4x_5 - 1\}$. We take a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R . Then, by using for example Mathematica, we see that a Gröbner basis for the ideal I contains the following polynomials: $(x_5 - 5)h_1(x_5)$, where $h_1(x_5)$ is a polynomial of x_5 of degree 80 with integer coefficients, and polynomials of the form

$$b_2x_2 + v_2(x_5), \quad b_3x_3 + v_3(x_5), \quad b_4x_4 + v_4(x_5) \quad (30)$$

where b_2, b_3, b_4 are integers and $v_2(x_5), v_3(x_5), v_4(x_5)$ are polynomials of x_5 with degree 80 of integer coefficients. For the case when $x_5 - 5 = 0$, we consider ideals I_1 of the polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5]$ generated by $\{f_1, f_2, f_3, f_4, y, x_2x_3x_4x_5 - 1, x_5 - 5\}$. Then, by taking a lexicographic order $>$ with $y > x_2 > x_3 > x_4 > x_5$ for a monomial ordering on R , we obtain a Gröbner basis for the ideals I_1 that contains polynomials $\{x_2 - 2, x_3 - 3, x_4 - 4, x_5 - 5\}$. This solution corresponds to the Kähler Einstein metric. For the case $h_1(x_5) = 0$, we see that there are 18 positive solutions for x_5 . After substituting these values in the equations $b_2x_2 + v_2(x_5) = 0$, $b_3x_3 + v_3(x_5) = 0$, $b_4x_4 + v_4(x_5) = 0$, we see that there are 5 cases that all values for x_2, x_3 and x_4 are positive.

Thus we get:

Proposition 7. *The generalized flag manifold $M = G/H = E_8/U(1) \times SU(4) \times SU(5)$ admits (up to a scale) precisely five non-Kähler E_8 -invariant Einstein metrics. These E_8 -invariant Einstein metrics $g = (x_1, x_2, x_3, x_4, x_5)$ are given approximately by*

- (1) $x_1 = 1, \quad x_2 \approx 1.0213742, \quad x_3 \approx 0.54600746, \quad x_4 \approx 1.0535169, \quad x_5 \approx 1.1087938,$
- (2) $x_1 = 1, \quad x_2 \approx 1.0373227, \quad x_3 \approx 1.0471761, \quad x_4 \approx 1.0308150, \quad x_5 \approx 0.29861996,$
- (3) $x_1 = 1, \quad x_2 \approx 0.59978523, \quad x_3 \approx 1.0837088, \quad x_4 \approx 0.90182312, \quad x_5 \approx 1.2229122,$
- (4) $x_1 = 1, \quad x_2 \approx 0.72071315, \quad x_3 \approx 1.0254588, \quad x_4 \approx 0.47523403, \quad x_5 \approx 1.0709463,$
- (5) $x_1 = 1, \quad x_2 \approx 1.0829413, \quad x_3 \approx 1.0408835, \quad x_4 \approx 0.53261506, \quad x_5 \approx 1.1035115.$

and the Einstein constants λ are given by

- (1) $\lambda \approx 0.36550657, \quad (2) \lambda \approx 0.33727144, \quad (3) \lambda \approx 0.37877040, \quad (4) \lambda \approx 0.38698208, \quad (5) \lambda \approx 0.33939371.$

For any G -invariant Einstein metric $g = (x_1, x_2, x_3, x_4, x_5)$ on $M = G/H$, we consider the scale invariant given by $H_g = V_g^{1/d} S_g$, where $d = \sum_{i=1}^5 d_i$, S_g is the scalar curvature of g and V_g is the volume $V_g = \prod_{i=1}^5 x_i^{d_i}$ of the given metric g (cf. [AC3]). We compute the scale invariant H_g for invariant Einstein metrics above and we see that

- (1) $H_g \approx 68.7023, \quad (2) H_g \approx 68.4799, \quad (3) H_g \approx 68.8906, \quad (4) H_g \approx 68.6914, \quad (5) H_g \approx 68.7757$

respectively. Thus we conclude that these invariant Einstein metrics can not be isometric each other.

By normalizing Einstein constant $\lambda = 1$, we obtain:

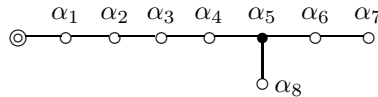
Theorem 2. *The generalized flag manifold $M = G/H = E_8/U(1) \times SU(4) \times SU(5)$ admits precisely five non-Kähler E_8 -invariant Einstein metrics up to isometry. These E_8 -invariant Einstein metrics $g = (x_1, x_2, x_3, x_4, x_5)$ are given approximately by*

- (1) $x_1 \approx 0.36550657, \quad x_2 \approx 0.37331898, \quad x_3 \approx 0.19956931, \quad x_4 \approx 0.38506736, \quad x_5 \approx 0.40527143,$
- (2) $x_1 \approx 0.33727144, \quad x_2 \approx 0.34985931, \quad x_3 \approx 0.35318260, \quad x_4 \approx 0.34766447, \quad x_5 \approx 0.10071598,$
- (3) $x_1 \approx 0.37877040, \quad x_2 \approx 0.22718089, \quad x_3 \approx 0.41047683, \quad x_4 \approx 0.34158391, \quad x_5 \approx 0.46320296,$
- (4) $x_1 \approx 0.38698208, \quad x_2 \approx 0.27890308, \quad x_3 \approx 0.39683418, \quad x_4 \approx 0.18390705, \quad x_5 \approx 0.41443703,$
- (5) $x_1 \approx 0.33939371, \quad x_2 \approx 0.36754348, \quad x_3 \approx 0.35326931, \quad x_4 \approx 0.18076620, \quad x_5 \approx 0.37452488.$

5. HOMOGENEOUS EINSTEIN METRICS ON $E_8/U(1) \times SU(2) \times SU(3) \times SU(5)$

5.1. The construction of the homogeneous Einstein equation on $E_8/U(1) \times SU(2) \times SU(3) \times SU(5)$.

We will exam now the case (F). We consider again the Lie group $G = E_8$ and we set $\Pi_M = \{\alpha_5\}$ and $\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8\}$. This choice gives rise to the following (extended) painted Dynkin diagram



It defines the flag manifold $M = G/H = E_8/U(1) \times SU(2) \times SU(3) \times SU(5)$. From Lemma 2 and since we have $\text{ht}(\alpha_5) = 6$, it follows that $N = 6 = q$, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$. Thus we can find a pair (Π, Π_0) for $\mathfrak{g} = \mathfrak{e}_8$, which has an irreducible decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$ as $\text{Ad}(H)$ -modules, where \mathfrak{h}_0 is the center of \mathfrak{h} and $\mathfrak{h}_1 = \mathfrak{su}(2)$, $\mathfrak{h}_2 = \mathfrak{su}(3)$, $\mathfrak{h}_3 = \mathfrak{su}(5)$. Note that $d_0 = \dim \mathfrak{h}_0 = 1$, $d_1 = \dim \mathfrak{h}_1 = 3$, $d_2 = \dim \mathfrak{h}_2 = 8$ and $d_3 = \dim \mathfrak{h}_3 = 24$. Also from Lemma 2, we obtain that $d_4 = \dim \mathfrak{m}_1 = 60$, $d_5 = \dim \mathfrak{m}_2 = 60$, $d_6 = \dim \mathfrak{m}_3 = 40$, $d_7 = \dim \mathfrak{m}_4 = 30$, $d_8 = \dim \mathfrak{m}_5 = 12$ and $d_9 = \dim \mathfrak{m}_6 = 10$.

Proposition 8. *In the decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$ the ideals \mathfrak{h}_1 and \mathfrak{h}_2 can be taken such that $[\mathfrak{h}_1, \mathfrak{m}_6] = [\mathfrak{h}_2, \mathfrak{m}_6] = \{0\}$.*

Proof. Since $\mathfrak{h}_1 = \mathfrak{su}(2)$, and $\mathfrak{h}_2 = \mathfrak{su}(3)$, we can assume that $\mathfrak{h}_1 \neq \{0\}$ and $\mathfrak{h}_2 \neq \{0\}$. Note that there is only a simple root $\alpha_{j_0} = \alpha_8$ with $(\alpha_{j_0}, \tilde{\alpha}) \neq 0$ and thus we can take the ideals \mathfrak{h}_1 and \mathfrak{h}_2 such that $[\mathfrak{h}_1^{\mathbb{C}}, E_{\tilde{\alpha}}] = [\mathfrak{h}_2^{\mathbb{C}}, E_{\tilde{\alpha}}] = \{0\}$. Since $\mathfrak{n}_6 = [\mathfrak{h}^{\mathbb{C}}, E_{\tilde{\alpha}}]$, we have that $[\mathfrak{h}_1^{\mathbb{C}}, \mathfrak{n}_6] = [\mathfrak{h}_1^{\mathbb{C}}, [\mathfrak{h}^{\mathbb{C}}, E_{\tilde{\alpha}}]] \subset [[\mathfrak{h}_1^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}], E_{\tilde{\alpha}}] + [\mathfrak{h}^{\mathbb{C}}, [\mathfrak{h}_1^{\mathbb{C}}, E_{\tilde{\alpha}}]] = \{0\}$. By the definition of \mathfrak{m}_6 , we get the result. Similar for \mathfrak{h}_2 . \square

Now, we consider an E_8 -invariant Riemannian metric $(\ , \)$ on $G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ given by

$$(\ , \) = x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5} + x_6 \cdot B|_{\mathfrak{m}_6}, \quad (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}_+^6. \quad (31)$$

Proposition 9. *The components r_i of the Ricci tensor r for the G -invariant metric $(\ , \)$ on $G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ defined by (31), are given as follows:*

$$\left\{ \begin{array}{l} r_1 = \frac{1}{2x_1} - \frac{1}{2d_1} \left[\frac{2}{11} \frac{x_2}{x_1^2} + \frac{1}{2d_1} \left[\frac{1}{23} \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right) + \frac{1}{2d_1} \left[\frac{1}{34} \left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3} \right) \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2d_1} \left[\frac{1}{45} \left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4} \right) + \frac{1}{2d_1} \left[\frac{1}{56} \left(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5} \right) \right] \right] \right] \right. \\ r_2 = \frac{1}{2x_2} + \frac{1}{4d_2} \left[\frac{2}{11} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) - \frac{1}{2d_2} \left[\frac{4}{22} \frac{x_4}{x_2^2} + \frac{1}{2d_2} \left[\frac{2}{13} \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1} \right) \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2d_2} \left[\frac{2}{35} \left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3} \right) + \frac{1}{2d_2} \left[\frac{2}{46} \left(\frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} - \frac{x_6}{x_2x_4} \right) \right] \right] \right] \right. \\ r_3 = \frac{1}{2x_3} - \frac{1}{2d_3} \left[\frac{6}{33} \frac{x_6}{x_3^2} + \frac{1}{2d_3} \left[\frac{3}{12} \left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2} \right) + \frac{1}{2d_3} \left[\frac{3}{14} \left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} \right) \right. \right. \right. \\ \left. \left. \left. + \frac{1}{2d_3} \left[\frac{3}{25} \left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2} \right) \right] \right] \right] \right. \\ r_4 = \frac{1}{2x_4} + \frac{1}{4d_4} \left[\frac{4}{22} \left(\frac{x_4}{x_2^2} - \frac{2}{x_4} \right) + \frac{1}{2d_4} \left[\frac{4}{13} \left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1} \right) \right. \right. \\ \left. \left. + \frac{1}{2d_4} \left[\frac{4}{15} \left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4} \right) + \frac{1}{2d_4} \left[\frac{4}{26} \left(\frac{x_4}{x_2x_6} - \frac{x_2}{x_4x_6} - \frac{x_6}{x_2x_4} \right) \right] \right] \right] \right. \\ r_5 = \frac{1}{2x_5} + \frac{1}{2d_5} \left[\frac{5}{14} \left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} \right) + \frac{1}{2d_5} \left[\frac{5}{23} \left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} \right) \right. \right. \\ \left. \left. + \frac{1}{2d_5} \left[\frac{5}{16} \left(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5} \right) \right] \right] \right. \\ r_6 = \frac{1}{2x_6} + \frac{1}{4d_6} \left[\frac{6}{33} \left(\frac{x_6}{x_3^2} - \frac{2}{x_6} \right) + \frac{1}{2d_6} \left[\frac{6}{15} \left(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} \right) \right. \right. \\ \left. \left. + \frac{1}{2d_6} \left[\frac{6}{24} \left(\frac{x_6}{x_2x_4} - \frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} \right) \right] \right] \right. \end{array} \right. \quad (32)$$

From Proposition 3, we known that the unique E_8 -invariant Kähler-Einstein metric on G/H is given by $B|_{\mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2} + 3 \cdot B|_{\mathfrak{m}_3} + 4 \cdot B|_{\mathfrak{m}_4} + 5 \cdot B|_{\mathfrak{m}_5} + 6 \cdot B|_{\mathfrak{m}_6}$. We use these parameters to obtain the following equations:

$$\begin{aligned} & \frac{1}{2} - \frac{1}{d_1} \left(\left[\frac{2}{11} \right] + \left[\frac{3}{12} \right] + \left[\frac{4}{13} \right] + \left[\frac{5}{14} \right] + \left[\frac{6}{15} \right] \right) = \frac{1}{4} + \frac{1}{d_2} \left(\frac{1}{4} \left[\frac{2}{11} \right] - \frac{1}{2} \left[\frac{3}{12} \right] - \frac{1}{2} \left[\frac{4}{22} \right] - \frac{1}{2} \left[\frac{5}{23} \right] - \frac{1}{2} \left[\frac{6}{24} \right] \right) \\ & = \frac{1}{6} + \frac{1}{d_3} \left(\frac{1}{3} \left[\frac{3}{12} \right] - \frac{1}{3} \left[\frac{4}{13} \right] - \frac{1}{3} \left[\frac{5}{23} \right] - \frac{1}{3} \left[\frac{6}{33} \right] \right) = \frac{1}{8} + \frac{1}{d_4} \left(\frac{1}{4} \left[\frac{4}{13} \right] - \frac{1}{4} \left[\frac{5}{14} \right] + \frac{1}{8} \left[\frac{4}{22} \right] - \frac{1}{4} \left[\frac{6}{24} \right] \right) \\ & = \frac{1}{10} + \frac{1}{d_5} \left(\frac{1}{5} \left[\frac{5}{14} \right] - \frac{1}{5} \left[\frac{6}{15} \right] + \frac{1}{5} \left[\frac{5}{23} \right] \right) = \frac{1}{12} + \frac{1}{d_6} \left(\frac{1}{6} \left[\frac{6}{15} \right] + \frac{1}{6} \left[\frac{6}{24} \right] + \frac{1}{12} \left[\frac{6}{33} \right] \right). \end{aligned} \quad (33)$$

5.2. Use of submersion. From equations (33) we obtain a system with five equations and nine unknowns, namely the triples

$$\begin{bmatrix} 2 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \end{bmatrix}, \begin{bmatrix} 5 \\ 14 \end{bmatrix}, \begin{bmatrix} 6 \\ 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 22 \end{bmatrix}, \begin{bmatrix} 5 \\ 23 \end{bmatrix}, \begin{bmatrix} 6 \\ 24 \end{bmatrix}, \begin{bmatrix} 6 \\ 33 \end{bmatrix}.$$

We put $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_6$, $\mathfrak{p}_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_5$, $\mathfrak{p}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_4$, $\mathfrak{p}_3 = \mathfrak{m}_3$ and $\mathfrak{q}_1 = \mathfrak{m}_6$. Then \mathfrak{k} is a subalgebra of \mathfrak{g} , and from Proposition 8 we conclude that \mathfrak{k}_1 is also a subalgebra of \mathfrak{g} . In particular, we have $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1 = \mathfrak{su}(2)$, and $\mathfrak{h}_2 = \mathfrak{su}(3)$. Also, for dimensional reasons it is $\mathfrak{k}_1 = \mathfrak{su}(6)$. Now, by using (13) we obtain the following inclusions:

$$\begin{aligned} [\mathfrak{p}_1, \mathfrak{p}_1] &\subset \mathfrak{p}_2 \oplus \mathfrak{k}, & [\mathfrak{p}_1, \mathfrak{p}_3] &\subset \mathfrak{p}_2, & [\mathfrak{p}_2, \mathfrak{p}_2] &\subset \mathfrak{p}_2 \oplus \mathfrak{k}, \\ [\mathfrak{p}_1, \mathfrak{p}_2] &\subset \mathfrak{p}_1 \oplus \mathfrak{p}_3, & [\mathfrak{p}_2, \mathfrak{p}_3] &\subset \mathfrak{p}_1, & [\mathfrak{p}_3, \mathfrak{p}_3] &\subset \mathfrak{k}. \end{aligned} \quad (34)$$

Thus we obtain an irreducible decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ as $\text{Ad}(K)$ -modules, which are mutually non-equivalent. Since $\mathfrak{h} \subset \mathfrak{k}$, we can determine the fibration $G/H \rightarrow G/K$ given by

$$E_8 / U(1) \times SU(2) \times SU(3) \times SU(5) \rightarrow E_8 / SU(6) \times SU(2) \times SU(3).$$

We consider a Riemannian submersion $\pi : (G/H, g) \rightarrow (G/K, \check{g})$ with totally geodesic fibers isometric to $(K/H, \hat{g})$.

Note that a G -invariant metric \check{g} on $G/K = E_8 / SU(6) \times SU(2) \times SU(3)$ is given by

$$\check{g} = y_1 \cdot B|_{\mathfrak{p}_1} + y_2 \cdot B|_{\mathfrak{p}_2} + y_3 \cdot B|_{\mathfrak{p}_3} \quad (y_1, y_2, y_3) \in \mathbb{R}_+^3, \quad (35)$$

a G -invariant metric g on $G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ is given by

$$g = y_1 \cdot B|_{\mathfrak{p}_1} + y_2 \cdot B|_{\mathfrak{p}_2} + y_3 \cdot B|_{\mathfrak{p}_3} + z_1 \cdot B|_{\mathfrak{q}_1}, \quad (y_1, y_2, y_3, z_1) \in \mathbb{R}_+^3 \quad (36)$$

and a K -invariant metric \hat{g} on $K/H \simeq SU(6)/U(1) \times SU(5)$ is given by

$$\hat{g} = z_1 \cdot B|_{\mathfrak{q}_1}, \quad z_1 \in \mathbb{R}_+. \quad (37)$$

Notice that the metric (36) can be written as the metric of the form (31):

$$g = y_1 \cdot B|_{\mathfrak{m}_1} + y_2 \cdot B|_{\mathfrak{m}_2} + y_3 \cdot B|_{\mathfrak{m}_3} + y_2 \cdot B|_{\mathfrak{m}_4} + y_1 \cdot B|_{\mathfrak{m}_5} + z_1 \cdot B|_{\mathfrak{m}_6}. \quad (38)$$

From (32) we obtain components r_i of the Ricci tensor r for the metric (38) on G/H as follows:

$$\left\{ \begin{aligned} r_1 &= \frac{1}{2y_1} - \frac{1}{2d_1} \left(\begin{bmatrix} 2 \\ 11 \end{bmatrix} + \begin{bmatrix} 1 \\ 45 \end{bmatrix} \right) \frac{y_2}{y_1^2} + \frac{1}{2d_1} \left(\begin{bmatrix} 1 \\ 23 \end{bmatrix} + \begin{bmatrix} 1 \\ 34 \end{bmatrix} \right) \left(\frac{y_1}{y_3 y_2} - \frac{y_3}{y_1 y_2} - \frac{y_2}{y_1 y_3} \right) - \frac{1}{2d_1} \begin{bmatrix} 1 \\ 56 \end{bmatrix} \frac{z_1}{y_1^2} \\ r_2 &= \frac{1}{2y_2} + \frac{1}{4d_2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) - \frac{1}{2d_2} \begin{bmatrix} 4 \\ 22 \end{bmatrix} \frac{1}{y_2} \\ &\quad + \frac{1}{2d_2} \left(\begin{bmatrix} 2 \\ 13 \end{bmatrix} + \begin{bmatrix} 2 \\ 35 \end{bmatrix} \right) \left(\frac{y_2}{y_3 y_1} - \frac{y_3}{y_2 y_1} - \frac{y_1}{y_2 y_3} \right) - \frac{1}{2d_2} \begin{bmatrix} 2 \\ 46 \end{bmatrix} \frac{z_1}{y_2^2} \\ r_3 &= \frac{1}{2y_3} - \frac{1}{2d_3} \begin{bmatrix} 6 \\ 33 \end{bmatrix} \frac{z_1}{y_3^2} + \frac{1}{2d_3} \left(\begin{bmatrix} 3 \\ 12 \end{bmatrix} + \begin{bmatrix} 3 \\ 14 \end{bmatrix} + \begin{bmatrix} 3 \\ 25 \end{bmatrix} \right) \left(\frac{y_3}{y_2 y_1} - \frac{y_2}{y_3 y_1} - \frac{y_1}{y_3 y_2} \right) \\ r_4 &= \frac{1}{2y_2} - \frac{1}{4d_4} \begin{bmatrix} 4 \\ 22 \end{bmatrix} \frac{1}{y_2} + \frac{1}{2d_4} \begin{bmatrix} 4 \\ 13 \end{bmatrix} \left(\frac{y_2}{y_1 y_3} - \frac{y_1}{y_3 y_2} - \frac{y_3}{y_2 y_1} \right) + \frac{1}{2d_4} \begin{bmatrix} 4 \\ 15 \end{bmatrix} \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) - \frac{1}{2d_4} \begin{bmatrix} 4 \\ 26 \end{bmatrix} \frac{z_1}{y_2^2} \\ r_5 &= \frac{1}{2y_1} - \frac{1}{2d_5} \begin{bmatrix} 5 \\ 14 \end{bmatrix} \frac{y_2}{y_1^2} + \frac{1}{2d_5} \begin{bmatrix} 5 \\ 23 \end{bmatrix} \left(\frac{y_1}{y_2 y_3} - \frac{y_2}{y_3 y_1} - \frac{y_3}{y_2 y_1} \right) - \frac{1}{2d_5} \begin{bmatrix} 5 \\ 16 \end{bmatrix} \frac{z_1}{y_1^2} \\ r_6 &= \frac{1}{2z_1} + \frac{1}{4d_6} \begin{bmatrix} 6 \\ 33 \end{bmatrix} \left(\frac{z_1}{y_3^2} - \frac{2}{z_1} \right) + \frac{1}{2d_6} \begin{bmatrix} 6 \\ 15 \end{bmatrix} \left(\frac{z_1}{y_1^2} - \frac{2}{z_1} \right) + \frac{1}{2d_6} \begin{bmatrix} 6 \\ 24 \end{bmatrix} \left(\frac{z_1}{y_2^2} - \frac{2}{z_1} \right). \end{aligned} \right. \quad (39)$$

Now we consider a G -invariant metric \check{g} on $G/K = E_8 / SU(6) \times SU(2) \times SU(3)$ is given by (35).

Lemma 3. *For an invariant metric \check{g} on $E_8 / SU(6) \times SU(2) \times SU(3)$ given by (35), the non-zero structure constants are the following (and their symmetries):*

$$\begin{bmatrix} 2 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \end{bmatrix}.$$

Proof. This is an immediate consequence of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ and relation (34). \square

We set $\check{d}_1 = \dim \mathfrak{n}_1 = 72$, $\check{d}_2 = \dim \mathfrak{n}_2 = 90$ and $\check{d}_3 = \dim \mathfrak{n}_3 = 40$.

Proposition 10. *The components of the Ricci tensor \check{r} of the invariant metric \check{g} on E_8/K defined by (35), are given as follows:*

$$\left\{ \begin{array}{l} \check{r}_1 = \frac{1}{2y_1} - \frac{1}{2\check{d}_1} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \frac{y_2}{y_1^2} + \frac{1}{2\check{d}_1} \left[\begin{smallmatrix} 1 \\ 23 \end{smallmatrix} \right] \left(\frac{y_1}{y_2 y_3} - \frac{y_2}{y_1 y_3} - \frac{y_3}{y_1 y_2} \right) \\ \check{r}_2 = \frac{1}{2y_2} + \frac{1}{4\check{d}_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) + \frac{1}{2\check{d}_2} \left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix} \right] \left(\frac{y_2}{y_1 y_3} - \frac{y_1}{y_2 y_3} - \frac{y_3}{y_1 y_2} \right) - \frac{1}{4\check{d}_2} \left[\begin{smallmatrix} 2 \\ 22 \end{smallmatrix} \right] \frac{1}{y_2} \\ \check{r}_3 = \frac{1}{2y_3} + \frac{1}{2\check{d}_3} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] \left(\frac{y_3}{y_1 y_2} - \frac{y_1}{y_2 y_3} - \frac{y_2}{y_1 y_3} \right). \end{array} \right. \quad (40)$$

Proof. We use Lemma 3 and we apply again Theorem 1. \square

From Lemma 1, by taking the horizontal part of r_1 and r_5 , and r_2 and r_3 in (39), we see that

$$\left\{ \begin{array}{l} \check{r}_1 = \frac{1}{2y_1} - \frac{1}{2d_1} \left(\left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 1 \\ 45 \end{smallmatrix} \right] \right) \frac{y_2}{y_1^2} + \frac{1}{2d_1} \left(\left[\begin{smallmatrix} 1 \\ 23 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 1 \\ 34 \end{smallmatrix} \right] \right) \left(\frac{y_1}{y_3 y_2} - \frac{y_3}{y_1 y_2} - \frac{y_2}{y_1 y_3} \right) \\ = \frac{1}{2y_1} - \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] \frac{y_2}{y_1^2} + \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \left(\frac{y_1}{y_2 y_3} - \frac{y_2}{y_3 y_1} - \frac{y_3}{y_2 y_1} \right) \\ \check{r}_2 = \frac{1}{2y_2} + \frac{1}{4d_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right) - \frac{1}{2d_2} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \frac{1}{y_2} + \frac{1}{2d_2} \left(\left[\begin{smallmatrix} 2 \\ 13 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2 \\ 35 \end{smallmatrix} \right] \right) \left(\frac{y_2}{y_3 y_1} - \frac{y_3}{y_2 y_1} - \frac{y_1}{y_2 y_3} \right) \\ = \frac{1}{2y_2} - \frac{1}{4d_4} \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] \frac{1}{y_2} + \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] \left(\frac{y_2}{y_1 y_3} - \frac{y_1}{y_3 y_2} - \frac{y_3}{y_2 y_1} \right) + \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 15 \end{smallmatrix} \right] \left(\frac{y_2}{y_1^2} - \frac{2}{y_2} \right). \end{array} \right. \quad (41)$$

Thus we obtain the following equations:

$$\left. \begin{array}{l} \frac{1}{2\check{d}_1} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] = \frac{1}{2d_1} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] + \frac{1}{2d_1} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] = \frac{1}{2d_5} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] \\ \frac{1}{2\check{d}_2} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] = \frac{1}{2d_2} \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] + \frac{1}{2d_2} \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] = \frac{1}{2d_4} \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] \\ \frac{1}{2\check{d}_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] = \frac{1}{4d_2} \left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] = \frac{1}{2d_4} \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right]. \end{array} \right\}. \quad (42)$$

From equations (42), we obtain that

$$\left[\begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right] = 4 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right], \left[\begin{smallmatrix} 4 \\ 13 \end{smallmatrix} \right] = 2 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right], \left[\begin{smallmatrix} 3 \\ 12 \end{smallmatrix} \right] = 3 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right]. \quad (43)$$

From equations (33) and (43), we see that

$$\left\{ \begin{array}{l} 60 - 24 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] - 4 \left[\begin{smallmatrix} 6 \\ 15 \end{smallmatrix} \right] + 2 \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] - 12 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] + 2 \left[\begin{smallmatrix} 6 \\ 24 \end{smallmatrix} \right] = 0 \\ 20 + 4 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] - 2 \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] - 8 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] - 2 \left[\begin{smallmatrix} 6 \\ 24 \end{smallmatrix} \right] + 2 \left[\begin{smallmatrix} 6 \\ 33 \end{smallmatrix} \right] = 0 \\ 10 + 2 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] - \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] - 4 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] + 2 \left[\begin{smallmatrix} 6 \\ 24 \end{smallmatrix} \right] - 2 \left[\begin{smallmatrix} 6 \\ 33 \end{smallmatrix} \right] = 0 \\ 6 - 6 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] + 4 \left[\begin{smallmatrix} 6 \\ 15 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 4 \\ 22 \end{smallmatrix} \right] - 2 \left[\begin{smallmatrix} 6 \\ 24 \end{smallmatrix} \right] = 0 \\ 2 + 2 \left[\begin{smallmatrix} 5 \\ 14 \end{smallmatrix} \right] - 4 \left[\begin{smallmatrix} 6 \\ 15 \end{smallmatrix} \right] + 2 \left[\begin{smallmatrix} 5 \\ 23 \end{smallmatrix} \right] - 2 \left[\begin{smallmatrix} 6 \\ 24 \end{smallmatrix} \right] - \left[\begin{smallmatrix} 6 \\ 33 \end{smallmatrix} \right] = 0. \end{array} \right. \quad (44)$$

Now, by solving equations (44), we obtain that

$$\begin{bmatrix} 5 \\ 14 \end{bmatrix} = 1 + \begin{bmatrix} 4 \\ 22 \end{bmatrix}/6, \quad \begin{bmatrix} 6 \\ 15 \end{bmatrix} = 1, \quad \begin{bmatrix} 5 \\ 23 \end{bmatrix} = 3 - \begin{bmatrix} 4 \\ 22 \end{bmatrix}/6, \quad \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 2, \quad \begin{bmatrix} 6 \\ 33 \end{bmatrix} = 2. \quad (45)$$

5.2.1. The contribution of the twistor fibration. For the computation of the triples $\begin{bmatrix} 4 \\ 22 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 24 \end{bmatrix}$ we use the twistor fibration which admits any flag manifold $M = G/H$ of a compact (semi)-simple Lie group G , over an irreducible symmetric space G/L of compact type ([BuR, pp. 43-44]). This method was initially applied in [AC3].

We set $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_6$ and $\mathfrak{p} = \mathfrak{m}_1 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_5$. Then, in view of the inclusions given by (13) we conclude that $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$. Let L be the connected Lie subgroup of G with Lie algebra \mathfrak{l} . Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ is a reductive decomposition of G/L , and from the latter relations it follows that G/L is a locally symmetric space. In particular, since $G = E_8$ is a simply connected Lie group, G/L is also simply connected and thus it is a symmetric space. Because G is simple (and compact), G/L is an irreducible symmetric space (of compact type). In our case we have that $\dim \mathfrak{l} = 136$, thus it must be $G/L = E_8/E_7 \times \mathrm{SU}(2)$, since $\dim G/L = \dim G - \dim L = 278 - 136 = 112 = \dim \mathfrak{p}$. Since $\mathfrak{h} \subset \mathfrak{l}$ it follows that $H \subset L$, and thus we can determine the fibration $L/H \rightarrow G/H \xrightarrow{\pi} G/L$, explicitly given as follows:

$$E_7 \times \mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) \xrightarrow{\pi} E_8/E_7 \times \mathrm{SU}(2).$$

We observe that on the fiber L/H , the Lie group L does not act (almost) effectively, that is H contains some non-trivial normal subgroups of L . Let L' the normal subgroup of L which acts effectively on L/H with isotropy subgroup H' . Then $L/H = L'/H'$, that is

$$L/H = E_7 \times \mathrm{SU}(2)/\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(5) = E_7/\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5) = L'/H'.$$

The fiber L'/H' is a flag manifold with three isotropy summands ([Kim]): Let $\mathfrak{l}' = \mathfrak{h}' \oplus \mathfrak{f}$ be a reductive decomposition of \mathfrak{l}' with respect to B_{E_7} , the negative of the Killing form of E_7 . Then $T_{o'}(L'/H') = \mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3$, where $\mathfrak{f}_1 = \mathfrak{m}_2$, $\mathfrak{f}_2 = \mathfrak{m}_4$, and $\mathfrak{f}_3 = \mathfrak{m}_6$. We set $D_1 = \dim \mathfrak{f}_1 = 60$, $D_2 = \dim \mathfrak{f}_2 = 30$ and $D_3 = \dim \mathfrak{f}_3 = 10$ and we consider E_7 -invariant metrics on $E_7/\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, of the form

$$g_{\mathfrak{f}} = w_1 \cdot B_{E_7}|_{\mathfrak{f}_1} + w_2 \cdot B_{E_7}|_{\mathfrak{f}_2} + w_3 \cdot B_{E_7}|_{\mathfrak{f}_3}, \quad (w_1, w_2, w_3) \in \mathbb{R}_+^3. \quad (46)$$

Lemma 4. *For a L' -invariant metric $g_{\mathfrak{f}}$ on the fiber L'/H' given by (46), the non-zero structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}_{\mathfrak{f}}$ are $\begin{bmatrix} 2 \\ 11 \end{bmatrix}_{\mathfrak{f}}$ and $\begin{bmatrix} 3 \\ 12 \end{bmatrix}_{\mathfrak{f}}$ (and their symmetries).*

Proof. This result follows from the inclusions $[\mathfrak{f}_1, \mathfrak{f}_1] \subset \mathfrak{h}' \oplus \mathfrak{f}_2$, $[\mathfrak{f}_1, \mathfrak{f}_2] \subset \mathfrak{f}_1 \oplus \mathfrak{f}_3$, $[\mathfrak{f}_1, \mathfrak{f}_3] \subset \mathfrak{f}_2$, $[\mathfrak{f}_2, \mathfrak{f}_2] \subset \mathfrak{h}'$, $[\mathfrak{f}_2, \mathfrak{f}_3] \subset \mathfrak{f}_1$, and $[\mathfrak{f}_3, \mathfrak{f}_3] \subset \mathfrak{h}'$, which are easily obtained from relations given in (13). \square

Let R_i be the components of the Ricci tensor $\mathrm{Ric}_{g_{\mathfrak{f}}}$ for the E_7 -invariant metric $g_{\mathfrak{f}}$ on the fiber $L'/H' = E_7/\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$, defined by (46). Then, in view of Lemma 4 and by applying Theorem 1, (2), we obtain the following forms for the components R_i .

Proposition 11. *The components R_i of the Ricci tensor for an E_7 -invariant metric $g_{\mathfrak{f}}$ on the fiber $L'/H' = E_7/\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ defined by (46), are given as follows:*

$$\begin{cases} R_1 = \frac{1}{2w_1} - \frac{1}{2D_1} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \frac{w_2}{w_1^2} + \frac{1}{2D_1} \begin{bmatrix} 1 \\ 23 \end{bmatrix} \left(\frac{w_1}{w_2 w_3} - \frac{w_2}{w_1 w_3} - \frac{w_3}{w_1 w_2} \right) \\ R_2 = \frac{1}{2w_2} + \frac{1}{4D_2} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \left(\frac{w_2}{w_1^2} - \frac{2}{w_2} \right) + \frac{1}{2D_2} \begin{bmatrix} 2 \\ 13 \end{bmatrix} \left(\frac{w_2}{w_1 w_3} - \frac{w_1}{w_2 w_3} - \frac{w_3}{w_1 w_2} \right) \\ R_3 = \frac{1}{2w_3} + \frac{1}{2D_3} \begin{bmatrix} 3 \\ 12 \end{bmatrix} \left(\frac{w_3}{w_1 w_2} - \frac{w_1}{w_2 w_3} - \frac{w_2}{w_1 w_3} \right) \end{cases} \quad (47)$$

From Proposition 3 we know that $E_7/\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(5)$ admits a unique Kähler-Einstein metric, explicitly given by $1 \cdot B_{E_7}|_{\mathfrak{f}_1} + 2 \cdot B_{E_7}|_{\mathfrak{f}_2} + 3 \cdot B_{E_7}|_{\mathfrak{f}_3}$. Thus, by solving the system $\{R_1 - R_2 = 0, R_2 - R_3 = 0\}$, we obtain the values $\begin{bmatrix} 2 \\ 11 \end{bmatrix}_{\mathfrak{f}} = 10$ and $\begin{bmatrix} 3 \\ 12 \end{bmatrix}_{\mathfrak{f}} = 10/3$.

Since $L' = E_7$ is a simple Lie subgroup of E_8 there is a positive number c , such that $B_{E_7} = c \cdot B_{E_8}$, where $B_{E_8} = B$ is the Killing form of E_8 . In particular it is $c = B_{E_7}/B_{E_8} = 3/5$ (cf. [Brb]). Then, by applying an

easy computation based on the definition of the structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ we obtain that $\begin{bmatrix} 4 \\ 22 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 24 \end{bmatrix}$ are given as follows (see for example [AC3, Lemma 1]):

$$\begin{bmatrix} 4 \\ 22 \end{bmatrix} = c \cdot \begin{bmatrix} 2 \\ 11 \end{bmatrix}_f = 3/5 \cdot 10 = 6, \quad \begin{bmatrix} 6 \\ 24 \end{bmatrix} = c \cdot \begin{bmatrix} 3 \\ 12 \end{bmatrix}_f = 3/5 \cdot 10/3 = 2.$$

By substituting the values $\begin{bmatrix} 4 \\ 22 \end{bmatrix} = 6$ into equations (45), we get the explicit values of all non-zero triples for $E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ with respect to the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$.

Proposition 12. *For the E_8 -invariant metric (\cdot, \cdot) on $M = G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$ given by (31), the non-zero structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ are given as follows:*

$$\begin{bmatrix} 2 \\ 11 \end{bmatrix} = 8, \quad \begin{bmatrix} 3 \\ 12 \end{bmatrix} = 6, \quad \begin{bmatrix} 4 \\ 13 \end{bmatrix} = 4, \quad \begin{bmatrix} 5 \\ 14 \end{bmatrix} = 2, \quad \begin{bmatrix} 6 \\ 15 \end{bmatrix} = 1, \quad \begin{bmatrix} 4 \\ 22 \end{bmatrix} = 6, \quad \begin{bmatrix} 5 \\ 23 \end{bmatrix} = 2, \quad \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 2, \quad \begin{bmatrix} 6 \\ 33 \end{bmatrix} = 2.$$

5.2.2. Solutions of the homogeneous Einstein equation. By using Proposition 12 and the dimensions $d_i = \dim_{\mathbb{R}} \mathfrak{m}_i$ presented in §5.1, the components r_i ($1 \leq i \leq 6$) of the Ricci tensor are completely determined by equation (32). In particular, a G -invariant metric $(\cdot, \cdot) = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}_+^6$ on $G/H = E_8 / U(1) \times SU(2) \times SU(3) \times SU(5)$, is an Einstein metric, if and only if it is a positive real solution of the following system

$$\left\{ r_1 - r_2 = 0, \quad r_2 - r_3 = 0, \quad r_3 - r_4 = 0, \quad r_4 - r_5 = 0, \quad r_5 - r_6 = 0 \right\}, \quad (48)$$

where the components r_i are given as follows:

$$\left\{ \begin{aligned} r_1 &= \frac{1}{2x_1} - \frac{x_2}{15x_1^2} + \frac{1}{20} \left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} \right) + \frac{1}{30} \left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3} \right) \\ &\quad + \frac{1}{60} \left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4} \right) + \frac{1}{120} \left(\frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} - \frac{x_6}{x_1x_5} \right) \\ r_2 &= \frac{1}{2x_2} + \frac{1}{30} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) - \frac{1}{20} \frac{x_4}{x_2^2} + \frac{1}{20} \left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2} \right) + \frac{1}{60} \left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3} \right) \\ &\quad + \frac{1}{60} \left(\frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} - \frac{x_6}{x_2x_4} \right) \\ r_3 &= \frac{1}{2x_3} - \frac{1}{40} \frac{x_6}{x_3^2} + \frac{3}{40} \left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_3x_2} \right) + \frac{1}{20} \left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3} \right) \\ &\quad + \frac{1}{40} \left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2} \right) \\ r_4 &= \frac{1}{2x_4} + \frac{1}{20} \left(\frac{x_4}{x_2^2} - \frac{2}{x_4} \right) + \frac{1}{15} \left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} \right) + \frac{1}{30} \left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4} \right) \\ &\quad + \frac{1}{30} \left(\frac{x_4}{x_2x_6} - \frac{x_2}{x_4x_6} - \frac{x_6}{x_2x_4} \right) \\ r_5 &= \frac{1}{2x_5} + \frac{1}{12} \left(\frac{x_5}{x_1x_4} - \frac{x_4}{x_4x_5} - \frac{x_4}{x_1x_5} \right) + \frac{1}{12} \left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} \right) + \frac{1}{24} \left(\frac{x_5}{x_1x_6} - \frac{x_1}{x_5x_6} - \frac{x_6}{x_1x_5} \right) \\ r_6 &= \frac{1}{2x_6} + \frac{1}{20} \left(\frac{x_6}{x_3^2} - \frac{2}{x_6} \right) + \frac{1}{20} \left(\frac{x_6}{x_1x_5} - \frac{x_1}{x_5x_6} - \frac{x_5}{x_1x_6} \right) + \frac{1}{10} \left(\frac{x_6}{x_2x_4} - \frac{x_2}{x_4x_6} - \frac{x_4}{x_2x_6} \right). \end{aligned} \right. \quad (49)$$

We normalize our equations by setting $x_1 = 1$. We see that the system of equations (48) reduces to the following system of polynomial equations:

$$\left\{ \begin{aligned} f_1 &= -6x_3x_4^2x_5x_6 + 2x_2^3(x_4(1+6x_5)x_6 + x_3(x_5+6x_4x_5x_6)) - 2x_2(x_3^2x_4x_6 + x_4x_5(6+x_5)x_6 \\ &\quad + x_3x_5(x_4^2 - 26x_4x_6 + x_6^2)) + x_2^2(4x_3^2x_5x_6 + 4(-1+x_4^2)x_5x_6 + x_3(2x_4^2x_6 + 2(-1+x_5^2)x_6 \\ &\quad + x_4(-1+x_5^2 - 60x_5x_6 + x_6^2))) = 0 \\ f_2 &= -6x_3^2x_4^2x_5x_6 + 3x_2^2x_5x_6(-2x_3^3 + 2x_3(1-10x_4+x_4^2) + x_4x_6) + x_2^3x_3(5x_4(1+3x_5)x_6 \\ &\quad + 2x_3(x_5+2x_4x_5x_6)) + x_2x_3(x_4x_5(3+x_5)x_6 - 5x_3^2x_4(1+3x_5)x_6 - 2x_3x_5(x_4^2 - 26x_4x_6 + x_6^2)) = 0 \\ f_3 &= -6x_3^2x_4^2x_5x_6 + x_2^2x_6(14x_3^3x_5 + 2x_3(1+30x_4-7x_4^2)x_5 - 4x_3^2(-1+x_4^2+12x_5-x_5^2) - 3x_4x_5x_6) \\ &\quad + x_2^3x_3(4x_3x_5 - 3x_4(1+3x_5)x_6) + x_2x_3(-3x_4x_5(3+x_5)x_6 + 3x_3^2x_4(1+3x_5)x_6 + 4x_3x_5(-x_4^2+x_6^2)) = 0 \\ f_4 &= 6x_3x_4^2x_5x_6 + x_2^2(-4x_3x_5 + 10x_4x_6) + 2x_2(5x_3^2x_4x_6 - 5x_4x_5^2x_6 + 2x_3x_5(x_4^2 - x_6^2)) + x_2^2(-8x_3^2x_5x_6 \\ &\quad + 8(-1+x_4^2)x_5x_6 + x_3(14x_4^2x_6 + 2(3+24x_5-7x_5^2)x_6 - 5x_4(-1+x_5^2+12x_6-x_6^2))) = 0 \\ f_5 &= 2x_2^2x_3(6x_3x_5 - 5x_4x_6) - 2x_3(5x_3^2x_4x_6 - 5x_4x_5^2x_6 + 6x_3x_5(-x_4^2+x_6^2)) \\ &\quad + x_2(-6x_4x_5x_6^2 + x_3^2(-10x_4^2x_6 + 10(-1+x_5^2)x_6 + x_4(1-48x_5+11x_5^2+60x_6-11x_6^2))) = 0. \end{aligned} \right. \quad (50)$$

We need now to find non-zero solutions of equations (50). By following a similar approach like case (E), i.e., by considering a polynomial ring $R = \mathbb{Q}[y, x_2, x_3, x_4, x_5, x_6]$ and an ideal I generated by

$$\{f_1, f_2, f_3, f_4, f_5, y x_2 x_3 x_4 x_5 x_6 - 1\},$$

then we see that is very difficult to compute a Gröbner basis for the ideal I . For this case we use the software package HOM4PS-2.0, which is based on the homotopy continuation method for solving polynomial systems (see [LeLT]) and enable us to obtain explicitly all positive real solutions of system (50). We present the following result:

Proposition 13. *The generalized flag manifold $M = G/H = E_8 / \text{SU}(5) \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ admits (up to a scale) precisely four non-Kähler E_8 -invariant Einstein metrics. These E_8 -invariant Einstein metrics $g = (x_1, x_2, x_3, x_4, x_5, x_6)$ are given approximately by*

- (1) $x_1 = 1, \quad x_2 \approx 0.954875, \quad x_3 \approx 0.965321, \quad x_4 \approx 1.00534, \quad x_5 \approx 0.290091, \quad x_6 \approx 1.01965.$
- (2) $x_1 = 1, \quad x_2 \approx 0.986536, \quad x_3 \approx 0.636844, \quad x_4 \approx 1.06853, \quad x_5 \approx 1.13323, \quad x_6 \approx 0.921127.$
- (3) $x_1 = 1, \quad x_2 \approx 0.90422, \quad x_3 \approx 0.778283, \quad x_4 \approx 0.927483, \quad x_5 \approx 1.03408, \quad x_6 \approx 0.359949.$
- (4) $x_1 = 1, \quad x_2 \approx 0.82308, \quad x_3 \approx 1.14673, \quad x_4 \approx 1.17377, \quad x_5 \approx 1.42664, \quad x_6 \approx 1.46519.$

and the Einstein constants λ are given by

- (1) $\lambda \approx 67.805543,$ (2) $\lambda \approx 0.348602829,$ (3) $\lambda \approx 68.228353,$ (4) $\lambda \approx 0.313933143,$

respectively.

Similarly with case (E), for any G -invariant Einstein metric $g = (x_1, x_2, x_3, x_4, x_5, x_6)$ on M we consider the scale invariant $H_g = V_g^{1/d} S_g$, where $d = \sum_{i=1}^6 d_i$, S_g is the scalar curvature of g and V_g is the volume $V_g = \prod_{i=1}^6 x_i^{d_i}$ of the given metric g . We compute the scale invariant H_g for invariant Einstein metrics above and we see that

- (1) $H_g \approx 67.805543,$ (2) $H_g \approx 68.468503,$ (3) $H_g \approx 68.228353,$ (4) $H_g \approx 68.685589$

respectively. Since we get different values we conclude that these invariant Einstein metrics can not be isometric each other.

By normalizing Einstein constant $\lambda = 1$, we conclude that

Theorem 3. *The generalized flag manifold $M = G/H = E_8 / \text{SU}(5) \times \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ admits precisely four non-Kähler E_8 -invariant Einstein metrics up to isometry. These E_8 -invariant Einstein metrics $g = (x_1, x_2, x_3, x_4, x_5, x_6)$ are given approximately by*

- (1) $x_1 \approx 0.349296 \quad x_2 \approx 0.333534, \quad x_3 \approx 0.337183, \quad x_4 \approx 0.35116, \quad x_5 \approx 0.101328, \quad x_6 \approx 0.356159,$
- (2) $x_1 \approx 0.348603 \quad x_2 \approx 0.343909, \quad x_3 \approx 0.222006, \quad x_4 \approx 0.372492, \quad x_5 \approx 0.395047, \quad x_6 \approx 0.321107,$
- (3) $x_1 \approx 0.367518, \quad x_2 \approx 0.332318, \quad x_3 \approx 0.286033, \quad x_4 \approx 0.340867, \quad x_5 \approx 0.380043, \quad x_6 \approx 0.132288,$
- (4) $x_1 \approx 0.313933, \quad x_2 \approx 0.258393, \quad x_3 \approx 0.359988 \quad x_4 \approx 0.368484, \quad x_5 \approx 0.44787 \quad x_6 \approx 0.459972.$

Main Theorem in Introduction is now a consequence of Theorems 2 and 3, and the results stated in Table 1.

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